

THE GENERATION OF GAMMA DISTRIBUTED VARI-  
ATES AND AN INVESTIGATION OF A TREND TEST  
FOR THE GAMMA RENEWAL PROCESS

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# NAVAL POSTGRADUATE SCHOOL

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## THESIS

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AND AN INVESTIGATION OF  
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# ABSTRACT

In testing the hypothesis that there is no monotone trend in a gamma renewal process, the use of the statistic

$$Y_J = Y_{2J} / Y_{1J},$$

where

$$Y_{1J} = \sum_{i=1}^J X_i$$

and

$$Y_{2J} = \sum_{i=1}^J S_i,$$

is investigated. The mean and variance of  $Y_J$  is developed as a function of  $J$  and it is shown that  $Y_J$  is asymptotically normal as  $J \rightarrow \infty$  for the gamma renewal process. A high-speed, theoretically exact gamma pseudo-random variate generator is developed, tested and compared with other known techniques. The generator is then used to obtain the distribution of  $Y_J$  through digital computer simulation for small and moderate values of  $J$ .





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## I. INTRODUCTION

A renewal process is a model for a series of events which occur at random times; the times between each two successive events are independent and identically distributed positive random variables. The time between event  $i-1$  and event  $i$  is denoted by  $X_i$ ,  $i=1,2,\dots$ , while the random variable  $S_i$  represents the time to the  $i$ -th event. When zero is taken as the time of initiation of the series of events then

$$S_i = \sum_{n=1}^i X_n \quad (i=1,2,\dots).$$

An equivalent representation of a series of events is the counting function  $N(t)$  which represents the number of events occurring in the interval  $(0,t]$ . The equivalence of the two representations is expressed in the fundamental identity

$$N(t) < n \text{ if and only if } S_n > t \quad (n=1,2,\dots).$$

Thus  $P\{N(t) < n\} = P\{S_n > t\}$ , where the notation  $P\{\}$  represents the probability of the event within the brackets.

Two important characteristics of any series of events are the so-called renewal function  $M(t)$ , which is the expected number of events up to time  $t$ , and its derivative the rate function:

$$M(t) = E[N(t)],$$

$$\lambda(t) = \frac{d}{dt} M(t).$$





A stationary renewal process is one in which  $N(t)$  has stationary increments (see Cox and Lewis [6], Chapter 4, for details). In this case, the renewal function is

$$M(t) = t / u,$$

and  $\lambda(t) = 1/u$ , where  $u$  is the mean time between events.

For an arbitrary series of events there are many alternatives to the renewal process. Successive inter-event times may not be independent or may depend on the entire history of the process, or the distributions of the times between events or the increments of  $N(t)$  may change in some manner as the process continues. When these distributional changes involve the expected values of the inter-event times or the value of  $\lambda(t)$ , then a trend is said to exist in the process. The trend may be on the serial number of the event; for example, a radio set may tend to fail more frequently as a function of the total number of failures which have occurred. Alternatively, the trend may be essentially independent of the number of events and reflect basically the elapsed time  $t$ ; such a trend on time might be observed in the arrival of jobs at a computer facility over a 24-hour day where the job arrival rate depends mainly on the time of day and not on the number of jobs submitted. A trend on time is usually expressed through a non-linear time dependence in  $M(t)$ .

Trends may be increasing, decreasing, cyclic or some combination of the three. The various statistical procedures designed to test whether an arbitrary series of events has a trend of any type are quite specific as to the type of process and the trend model assumed. For a more complete discussion of tests for trend in series of events see Cox and Lewis [6], Cox [7] and Lewis [18].

The simplest form of renewal process is the Poisson process in which the inter-event times have the negative exponential distribution. A more general process is the



gamma renewal process in which the  $X_i$ 's are independent gamma random variables with density function

$$f(x) = \frac{1}{\Gamma(k)} \lambda^k x^{k-1} e^{-\lambda x} \quad (k > 0).$$

In this parameterization,  $\lambda$  is a scale parameter while  $k$  characterizes the distribution and is called the shape parameter. The function  $\Gamma(k)$  is the complete gamma function or integral

$$\Gamma(k) = \int_0^{\infty} x^{k-1} e^{-x} dx.$$

In testing for trend on serial number in the gamma renewal process the use of the statistic

$$Y_J = Y_{2J} / Y_{1J},$$

where

$$Y_{1J} = \sum_{i=1}^J X_i$$

and

$$Y_{2J} = \sum_{i=1}^J S_i$$

is investigated in this paper. A test for trend using  $Y_J$  is developed, and the distribution of  $Y_J$  is investigated through simulation for small values of  $J$  and through normal theory results as  $J \rightarrow \infty$ .



## II. THEORETICAL RESULTS

### A. BACKGROUND

Cox [5] proposed the use of the statistic

$$B = \sum_{i=1}^J t_i$$

as a test for trend in a Poisson process, where  $t_1, t_2, \dots, t_J$  are the normalized times to events in a fixed observation interval of length  $T$  and  $J$  is the number of events occurring in  $(0, T]$ , that is, the observed value of  $N(T)$ . Bartholomew [4] dealt with the statistic

$$m = \frac{1}{J} B$$

and developed results on the asymptotic relative efficiency of tests for trend based on this statistic against several different trend models. Lewis [18] showed that the test based on  $m$  is not a consistent test for stationary Poisson processes against stationary gamma renewal alternatives, although it had been used for that purpose.

The test based on the statistic  $B$  is a conditional test, the conditioning being on the observed value  $J$  of the random variable  $N(T)$ . This is because the test is designed to detect trend regardless of the rate of occurrence of events. The test arises from the non-homogeneous Poisson process with rate function

$$\lambda(t) = \exp\{a + bt\} = \lambda e^{bt}.$$



In this case it is desired to test whether  $b=0$ , that is whether there is a trend. It is possible to obtain the likelihood function  $L(\lambda, b)$  of the observations for this model and from Neyman-Pearson theory conclude that the test based on  $B$  is the uniformly most powerful conditional test

$$H_0 : b=0, \lambda > 0$$

$$H_1 : b \neq 0, \lambda > 0$$

for every value of  $T$ . For details, see Cox and Lewis [6], chapters 2 and 3.

For the foregoing model, the parameter  $\lambda$  is a nuisance parameter since it has no effect on the trend; it merely controls the base rate of the process. It is eliminated in the  $B$  test by conditioning on  $J = N(T)$ , this being a sufficient statistic for  $\lambda$  for any fixed value of  $b$ .

There is an analogous situation when testing for trend on serial number and the process is observed for a fixed number of intervals  $J$ . Here, the trend model is one of exponentially distributed intervals with

$$E[X_i] = \exp\{a + bi\} = \lambda e^{bi} \quad (i=1, 2, \dots).$$

The sufficient statistic for  $\lambda$  is now  $S_J = Y_{1J}$  so that the test is now conditioned on the observed value of  $Y_{1J}$  and the statistic analogous to  $B$  is

$$Y_{2J} = \sum_{i=1}^J S_i = \sum_{i=1}^J (J+1-i) X_i.$$

Asymptotically, there is no difference between conditioning on a fixed time interval and conditioning on a fixed number of events but the latter turns out to be more convenient for the gamma renewal process. In this case





performing a test for trend analogous to the Poisson process test based on  $B$  requires the determination of the conditional distribution of  $Y_{2J}$  given  $Y_{1J}$ .

## B. THE GAMMA RENEWAL PROCESS

In the gamma renewal process, it is both analytically and computationally simpler to test for trend on serial number over a fixed number of observations than to test for trend in time over a fixed time period. The computational advantage of the former test arises from the ease of computer simulation of a fixed number of events as compared to the difficulties inherent in continuous time simulation control. Analytically, the gamma renewal process has certain theoretical properties which simplify consideration of the serial trend test.

For the gamma renewal process with shape parameter  $k$  and the assumed trend model (Cox [7])

$$E[X_i] = \exp\{a + bi\} = \lambda e^{bi} \quad (i=1,2,\dots)$$

it is possible to set up the likelihood function  $L(k, b, \lambda)$ . Explicit results, however, cannot be obtained for estimating  $b$  or testing whether  $b=0$ ; it is therefore usual to use the Poisson test based on  $B$  in this case. The test will probably be relatively powerful for  $k$  near one but not when  $k$  is small as it uses very little information about the individual  $X_i$ 's. In fact, the test is based only on the centroid of the times to events and not on times between events.

Elimination of the nuisance parameter  $\lambda$  cannot be accomplished for the gamma renewal process by conditioning



on  $S_J$  as it could in the Poisson case, since  $S_J$  is not a sufficient statistic for  $\lambda$  for all values of  $b$ . However, since  $\lambda$  is a pure scale factor the ratio statistic

$$Y_J = Y_{2J} / Y_{1J}$$

is free of  $\lambda$  for any  $b$  and  $k$  and is thus a reasonable alternative statistic on which to base a test which is free of the scale factor.

The distributions of the two test statistics (conditional and ratio) are equivalent in the gamma case under the null hypothesis ( $b=0$ ) because of a result which characterizes the gamma distribution. Laha [16] and Lukacs [21] have established that, for a sequence of independent and identically distributed random variables  $X_1, X_2, \dots, X_n$ , the statistics

$$\sum_{i=1}^J (J+1-i) X_i / \sum_{i=1}^J X_i$$

and

$$\sum_{i=1}^J X_i$$

are independent if and only if the  $X_i$ 's have the gamma distribution. Thus,  $Y_J$  is independent of  $Y_{1J}$  for the gamma renewal process. Using this result,

$$\begin{aligned} P\{Y_J \leq z\} &= P\{Y_{2J} / Y_{1J} \leq z\} \\ &= P\{Y_{2J} / Y_{1J} \leq z \mid Y_{1J} = y\} \\ &= P\{Y_{2J} \leq zy \mid Y_{1J} = y\}, \end{aligned}$$



so that the conditional and ratio statistics have the same distribution. Furthermore, the gamma renewal process is the only one for which this is the case.

A further result of the above characterization and the probability relationship is that, if the joint distribution of  $Y_{1J}$  and  $Y_{2J}$  is asymptotically bivariate normal (a result which is proved in the following Section), then  $Y_J$  must also be asymptotically normal. This follows since the conditional distribution of  $Y_{2J}$  given  $Y_{1J}$  will be asymptotically normal under these conditions. Thus, although the asymptotic distribution of a ratio statistic such as  $Y_J$  can be very difficult to obtain in general, an expression can be found for the gamma case.

Trend tests using  $Y_J$  for other types of renewal process could and should be considered but the gamma process was chosen deliberately to take advantage of the above simplifying distributional results. The gamma process would probably not be the simplest case to consider under the trend alternative, however.

In the succeeding sections, results on the asymptotic distribution of  $Y_J$  and the mean and variance of  $Y_J$  for small values of  $J$  are developed. In addition, a description of a computer simulation experiment which was performed to obtain the distribution of  $Y_J$  for small values of  $J$  is described and its results presented. The power of the  $Y_J$  test for the gamma process has not been investigated nor has its power relative to nonparametric tests or tests using data



transformations been examined.

### C. ASYMPTOTIC DISTRIBUTIONS

The statistics  $Y_{1J}$  and  $Y_{2J}$  for a renewal process are defined by

$$Y_{1J} = \sum_{i=1}^J X_i,$$

$$Y_{2J} = \sum_{i=1}^J S_i = \sum_{i=1}^J (J+1-i) X_i,$$

where  $X_1, X_2, \dots, X_J$  are the process inter-event times and are assumed to have finite mean  $u$  and finite variance  $\sigma^2$ .

Obviously,  $Y_{1J}$  and  $Y_{2J}$  are dependent random variables with respective means and variances

$$\begin{aligned} m_1 &= E[Y_{1J}] \\ &= E\left[\sum_{i=1}^J X_i\right] \\ &= \sum_{i=1}^J E[X_i] \\ &= J E[X] \\ &= J u. \end{aligned}$$

$$\begin{aligned} s_1^2 &= \text{Var}[Y_{2J}] \\ &= \text{Var}\left[\sum_{i=1}^J X_i\right] \\ &= \sum_{i=1}^J \text{Var}[X_i] \\ &= J \text{Var}[X] \end{aligned}$$





$$= J \sigma^2.$$

$$\begin{aligned} m_2 &= E[Y_{2J}] \\ &= E\left[\sum_{i=1}^J (J+1-i) X_i\right] \\ &= \sum_{i=1}^J (J+1-i) E[X_i] \\ &= E[X] \cdot \left[\sum_{i=1}^J (J+1) - \sum_{i=1}^J i\right] \\ &= \frac{J(J+1)}{2} \mu. \end{aligned}$$

$$\begin{aligned} s_2^2 &= \text{Var}[Y_{2J}] \\ &= \text{Var}\left[\sum_{i=1}^J (J+1-i) X_i\right] \\ &= \sum_{i=1}^J (J+1-i)^2 \text{Var}[X_i] \\ &= \text{Var}[X] \sum_{i=1}^J i^2 \\ &= \frac{J(J+1)(2J+1)}{6} \sigma^2. \end{aligned}$$

To find the correlation coefficient for the statistics  $Y_{1J}$

and  $Y_{2J}$  it is necessary to determine the first joint moment:

$$\begin{aligned} m_{12} &= E[Y_{1J} Y_{2J}] \\ &= E\left[\sum_{i=1}^J X_i \cdot \sum_{i=1}^J (J+1-i) X_i\right] \\ &= E\left[\sum_{i=1}^J \sum_{k=1}^J (J+1-k) X_i X_k\right] \end{aligned}$$



$$\begin{aligned}
m_{12} &= E \left[ \sum_{i=1}^J (J+1-i) X_i^2 \right. \\
&\quad \left. + \sum_{i=1}^{J-1} \sum_{k=i+1}^J (2J+2-i-k) X_i X_k \right] \\
&= \sum_{i=1}^J (J+1-i) E[X_i^2] \\
&\quad + \sum_{i=1}^{J-1} \sum_{k=i+1}^J (2J+2-i-k) E[X_i X_k] \\
&= E[X^2] \sum_{i=1}^J i + \{E[X]\}^2 \sum_{i=1}^{J-1} \sum_{k=i+1}^J (i+k) .
\end{aligned}$$

After some simplification this becomes

$$m_{12} = \frac{J(J+1)}{2} E[X^2] + \frac{J(J-1)}{2} u^2 .$$

Then the covariance is

$$\begin{aligned}
u_{11} &= E[(Y_{1J} - m_1)(Y_{2J} - m_2)] \\
&= m_{12} - m_1 m_2 \\
&= \frac{J(J+1)}{2} \sigma^2
\end{aligned}$$

and the correlation coefficient is

$$\begin{aligned}
\rho &= u_{11} / s_1 s_2 \\
&= \frac{J(J+1)}{2} \sigma^2 / \sqrt{J} \cdot \sqrt{\frac{J(J+1)(2J+1)}{6}} \sigma^2 \\
&= \sqrt{\frac{6}{2}} \sqrt{\frac{J+1}{2J+1}} .
\end{aligned}$$

To obtain asymptotic results for  $Y_J$  as  $J \rightarrow \infty$ , it is necessary to show that the joint distribution of  $Y_{1J}$  and  $Y_{2J}$  is asymptotically a bivariate normal distribution. To do



this it is necessary to apply the Central Limit Theorem to the two-dimensional vector random variable

$$Z_J = (Y_{1J} \ Y_{2J}) .$$

Wald and Wolfowitz [29] have shown that  $Z_J$  is asymptotically normal if and only if

$$L_J = l_1 Y_{1J} + l_2 Y_{2J}$$

is also asymptotically normal for every choice of  $l_1$  and  $l_2$ .

To apply the Central Limit Theorem to  $L_J$ , define

$$\begin{aligned} L'_J &= L_J - E[L_J] \\ &= l_1 \sum_{i=1}^J X_i + l_2 \sum_{i=1}^J (J+1-i) X_i - E[L_J] \\ &= \sum_{i=1}^J [l_1 + l_2 (J+1-i)] (X_i - u) \\ &= \sum_{i=1}^J a_i (X_i - u) . \end{aligned}$$

$$a_i = l_1 + l_2 (J+1-i) .$$

$$F_i(v) = P\{a_i (X_i - u) \leq v\} .$$

$$\begin{aligned} s_l^2 &= \text{Var}[L_J] \\ &= \text{Var}[L'_J] \\ &= \text{Var}[X_i - u] \sum_{i=1}^J a_i^2 \end{aligned}$$



$$= \sigma^2 \sum_{i=1}^J a_i^2.$$

Then  $L_J$  will be asymptotically normal if and only if the Lindeberg condition is satisfied for every  $\epsilon > 0$  (Cramer [8]):

$$(2-1) \quad \lim_{J \rightarrow \infty} \frac{1}{S_J^2} \sum_{i=1}^J \left\{ \int_{|v| > \epsilon S_J} v^2 dF_i(v) \right\} = 0.$$

After making the substitution  $v = a_i z$  the integral becomes

$$\int_{|v| > \epsilon S_J} v^2 dF_i(v) = \int_{|a_i z| > \epsilon S_J} a_i^2 z^2 dF_i(a_i z).$$

This can be simplified by first noting that

$$\begin{aligned} F_i(a_i z) &= P\{a_i (X_i - u) \leq a_i z\} \\ &= P\{X_i - u \leq z\} \\ &= P\{X_i \leq z+u\} \\ &= F(z+u), \end{aligned}$$

where  $F(x)$  is the common distribution function of the  $X_i$ 's.

Secondly, if

$$|a_i z| > \epsilon S_J$$

then

$$\begin{aligned} |z| &> \epsilon S_J / |a_i| \\ &> \epsilon S_J / (|l_1| + |l_2| (J+1-i)) \end{aligned}$$





$$> e s_1 / (|l_1| + |l_2|^{(J+1-i)})$$

$$> e s_1 / (|l_1| + |l_2|^J).$$

We can thus assume, without loss of generality, that  $l_1 > 0$

and  $l_2 > 0$ . Then,

$$\int_{|v| > e s_1} v^2 dF_i(v) \leq a_i^2 \int_{|z| > e s_1 / (l_1 + l_2^J)} z^2 dF(z+u).$$

$$\begin{aligned} -\frac{1}{s^2} \sum_{i=1}^J \left\{ \int_{|v| > e s_1} v^2 dF_i(v) \right\} \\ \leq -\frac{1}{s^2} \left[ \sum_{i=1}^J a_i^2 \right] \int_{|z| > e s_1 / (l_1 + l_2^J)} z^2 dF(z+u) \end{aligned}$$

$$(2-2) \quad \leq -\frac{1}{\sigma^2} \int_{|z| > e s_1 / (l_1 + l_2^J)} z^2 dF(z+u).$$

Since by hypothesis the  $X_i$ 's have a finite second moment and since

$$\begin{aligned} \lim_{J \rightarrow \infty} e s_1 / (l_1 + l_2^J) \\ \geq (e/l_2) \lim_{J \rightarrow \infty} (s_1/J) \\ \geq (e\sigma/l_2) \lim_{J \rightarrow \infty} \frac{1}{J} \left\{ \sum_{i=1}^J a_i^2 \right\}^{1/2} \\ \geq (e\sigma/l_2) \lim_{J \rightarrow \infty} \left\{ \sum_{i=1}^J \frac{1}{J^2} [l_1 + l_2^{(J+1-i)}]^2 \right\}^{1/2} \\ \geq \infty \end{aligned}$$

the integral (2-2) approaches zero and so the Lindeberg condition (2-1) is satisfied.



It has thus been shown that  $Y_{1J}$  and  $Y_{2J}$  are asymptotically jointly normal. From the known properties of the two-dimensional normal distribution (Gnedenko [10]), the conditional distribution of  $Y_{2J}$  given  $Y_{1J}$  must also be asymptotically normal with mean and variance

$$\begin{aligned} E[Y_{2J} | Y_{1J} = y_1] &= m_2 + (\rho s_2 / s_1) (y_1 - m_1) \\ &= \frac{J(J+1)}{2} u + \frac{(J+1)}{2} (y_1 - J u) \\ &= \frac{J+1}{2} y_1. \end{aligned}$$

$$\begin{aligned} \text{Var}[Y_{2J} | Y_{1J} = y_1] &= s_2^2 (1 - \rho^2) \\ &= \frac{J(J+1)(2J+1)}{6} \sigma^2 \left(1 - \frac{6}{4} \frac{J+1}{2J+1}\right) \\ &= \frac{J(J-1)}{12} \sigma^2. \end{aligned}$$

From the results of the previous Section,  $Y_J$  is also asymptotically normal with mean and variance

$$\begin{aligned} E[Y_J] &= E[Y_{2J} / Y_{1J} | Y_{1J} = y_1] \\ &= \frac{J+1}{2}. \end{aligned}$$

$$\begin{aligned} \text{Var}[Y_J] &= \text{Var}[Y_{2J} / Y_{1J} | Y_{2J} = y_1] \\ &= \frac{J(J-1)}{12} (\sigma / y_1)^2. \end{aligned}$$

The foregoing result then suggests the following procedure to test for trend in a gamma renewal process. The



observed values of  $Y_{1J}$  and  $Y_{2J}$  can be used to accept or reject the null hypothesis of a trend-less process at any desired significance level  $\alpha$  using the conditional normal distribution given above. Rejection of the no-trend hypothesis will take place if the observed value of  $Y_J$  is greater than the  $1-\alpha/2$  quantile of the distribution of  $Y_J$  (for increasing trend) or if  $Y_J$  is less than the  $\alpha/2$  quantile (decreasing trend).

For the distribution of the sum of a series of independent exponential random variables to become approximately normal, however, requires over 20 terms; the gamma renewal process with shape parameter less than one requires several terms just to produce a single exponential variable. Thus, on the order of 100 events could be required to use the asymptotic test. For a small sample (roughly, one with fewer than  $20/k$  events) it may not be possible to use a normal approximation to the distribution of the test statistic.

If sample size is so small that the asymptotic result cannot be used, it is then necessary to develop the distribution of  $Y_J$  in order to use this test for trend. If the distribution cannot be determined analytically, computer simulation must be resorted to; it is still possible, however, to use the probability relationship between the distribution of  $Y_J$  and  $Y_{2J}$  given  $Y_{1J}$  to obtain the moments of  $Y_J$ .



#### D. THE MEAN AND VARIANCE OF $Y_J$ FOR SMALL SAMPLES

From the foregoing, it can be concluded that in a gamma renewal process

$$\begin{aligned} E[Y_J] &= E[Y_{2J} / Y_{1J}] \\ &= E[Y_{2J}] / E[Y_{1J}] \end{aligned}$$

since  $Y_{2J}/Y_{1J}$  is independent of  $Y_{1J}$ . Thus,

$$\begin{aligned} E[Y_J] &= m_2 / m_1 \\ &= \frac{J+1}{2}. \end{aligned}$$

Similarly,

$$\begin{aligned} E[Y_J^2] &= E[Y_{2J}^2 / Y_{1J}^2] \\ &= E[Y_{2J}^2] / E[Y_{1J}^2] \\ &= (s_2^2 + m_2^2) / (s_1^2 + m_1^2) \\ &= \frac{(1/6)J(J+1)(2J+1)\frac{\sigma^2}{J} + (1/4)J^2(J+1)\frac{u^2}{J^2}}{\frac{(J+1)}{12} \frac{2(2J+1)\frac{\sigma^2}{J} + 3J(J+1)u^2}{J^2}} \end{aligned}$$

Introducing the mean and variance of the gamma random variable,

$$u = k / \lambda$$





$$\sigma^2 = k / \lambda^2$$

$$\begin{aligned} E[Y_J^2] &= \frac{J+1}{12} \frac{2(2J+1)(k/\lambda^2)}{(k/\lambda^2)} + \frac{3J(J+1)(k^2/\lambda^2)}{J(k^2/\lambda^2)} \\ &= \frac{J+1}{12} \frac{2(2J+1)}{kJ+1} + \frac{3J(J+1)k}{kJ+1} \end{aligned}$$

The final result for the variance is then

$$\begin{aligned} \text{Var}[Y_J] &= E[Y_J^2] - \{E[Y_J]\}^2 \\ &= \frac{J-1}{12} \frac{J+1}{kJ+1} \end{aligned}$$

Note that for the Poisson process (for which  $k=1.0$ ),  $\text{Var}[Y_J] = \frac{J-1}{12}$ , which is also the result for the sum of  $J-1$  uniform random variables. When  $k \neq 1.0$  then, approximately,

$$\text{Var}[Y_J] \approx \frac{J-1}{12} C^2(X),$$

where  $C^2(X)$  is the coefficient of variation (which is  $1/k$  for the gamma random variable.) Thus, the greater variability in the inter-event times is reflected directly in the increased variability of the statistic  $Y_J$ . From the

point of view of a trend test, the variability of  $Y_J$  is increased to allow for the possible confusion of an actual trend with a sequence of long intervals in the stationary process. As mentioned previously, this indicates that more powerful tests for trend could be developed which use more information about the interval distribution than just the coefficient of variation.

The succeeding sections discuss determination of the



distribution of  $Y_J$  for small samples through digital computer simulation. Values of the predicted mean and variance of  $Y_J$  were calculated for each simulation run and the results are tabulated as follows (note that the mean does not depend on k):

J	MEAN	VARIANCE				
		k = .10	k = .25	k = .50	k = .75	k = 1.0
10	5.5	4.125	2.375	1.375	.971	.750
30	15.5	18.729	8.814	4.682	3.188	2.417
50	25.5	34.708	15.428	8.010	5.409	4.083
100	50.5	75.750	32.048	16.338	10.964	8.250



### III. GENERATING GAMMA VARIATES FOR SIMULATION

Although there were several known methods of generating gamma random variates (see Section III.E below), all were judged deficient for large-scale computer simulation use; some are statistically inadequate for large samples and all require too much computer time. A new method was therefore developed to overcome these difficulties.

#### A. DESCRIPTION OF ALGORITHM

Marsaglia's rectangle-wedge-tail methods for generating normal [14,23] and exponential [22] random variates were modified to produce unit gamma variates with shape parameter  $k$  less than one. The basic approach is to select with respective probabilities  $P_1, P_2, \dots, P_n$  one of a series of random variables with distribution functions  $F_1(x), F_2(x), \dots, F_n(x)$ . If  $x$  is the generated random number then

$$F(x) = P_1 F_1(x) + P_2 F_2(x) + \dots + P_n F_n(x).$$

Taking derivatives,

$$f(x) = P_1 f_1(x) + P_2 f_2(x) + \dots + P_n f_n(x).$$

The method is then equivalent to a geometric decomposition of the density function of the variate to be generated. For efficiency, the  $P_i$ 's and  $F_i$ 's are chosen so that most of the time the generated variate is selected from a simple



distribution (preferably a uniform distribution).

Gamma variates with any desired scale parameter are obtained by multiplying a unit gamma variate (that is, one with scale parameter one) by the reciprocal of the desired scale parameter  $\lambda$ . Thus, the method developed generates only unit gamma variates, which have density function

$$f(x) = \frac{1}{\Gamma(k)} x^{k-1} e^{-x}.$$

The unit variates can then be scaled to produce the desired result.

It is difficult to apply the decomposition technique to the gamma distribution because the density function has a singularity at the origin when the shape parameter is less than one. A low value of  $x$  (called  $x_1$ ) is thus selected as the starting point for the decomposition; values of the gamma variate less than  $x_1$  are generated with probability  $P_0$  by means of a series approximation to the inverse of the incomplete gamma function (see Subsection III.B.1.) A series of  $x_i$ 's is then determined by the relations

$$h_1 = .25 x_1$$

$$(3-1) \quad h_i = 2 h_{i-1}$$

$$x_i = x_{i-1} + h_{i-1} \quad (i=2, 3, \dots, I).$$

This procedure continues until the area under the density function curve from zero to  $x_I$  is greater than 0.999; the value  $N$  is then defined to be  $I-1$ . Dependent upon  $k$ , the parameter  $N$  varies between 20 and 100. Values of the





variate greater than  $x_{N+1}$  are generated by an iterative procedure (Subsection III.B.2) with probability  $p_{N+1}$ .

The density function has thus been decomposed into  $N+2$  regions: a "head" region below  $x_1$ , a "tail" region above  $x_{N+1}$  and a series of  $N$  vertical strips. Each strip is divided into a large rectangle and a small wedge, as shown in Figure 1. The probability for the strip as a whole can

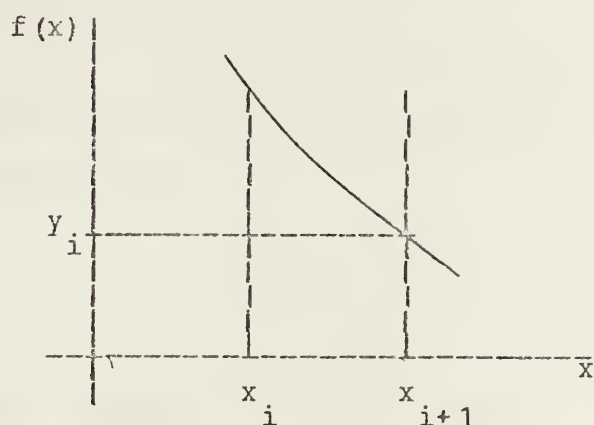


Figure 1. Decomposition of strip  $i$ .

be found by using the incomplete gamma function,

$$g(k; x) = \frac{1}{\Gamma(k)} \int_0^x \frac{y^{k-1}}{y} e^{-y} dy,$$

which is readily evaluated by an infinite series [1]

$$g(k; x) = \frac{1}{\Gamma(k)} \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+k}}{n! \Gamma(n+k)}$$

suitably truncated for computational purposes.

If  $P_i$  is the probability for the strip as a whole,  $p_i$  the probability of the rectangle and  $q_i$  the probability of



the wedge then the following relations exist:

$$P_i = g(k; x_{i+1}) - g(k; x_i),$$

$$p_i = h_i f(x_{i+1}),$$

$$q_i = P_i - p_i.$$

For small values of  $i$ ,  $q_i \gg p_i$  because of the extreme steepness of  $f(x)$  near the origin; this accounts for the increasing values of  $h_i$  in (3-1). For  $i$  greater than 5, this relationship reverses so that most of the area under the  $f(x)$  curve lies within the rectangles. In fact, it was found empirically that

$$\sum_{i=1}^N p_i \doteq 0.72$$

for most values of  $k$  in the range 0.1 to 1.0. Thus, the gamma random variable can be sampled from a uniform distribution over 70 per cent of the time. Presumably, an increase in this probability would result from choosing each  $h_i$  optimally, but the gains to be made are small.

## 1. Binary Search Scheme

With the decomposition complete, there remain two problems: efficient selection of one of some 100 sampling distributions and methods for sampling from each of the four different types of distribution (head, tail, rectangle and wedge).

To formalize the selection problem, suppose that events  $f_1, f_2, \dots, f_M$  are to be selected with probabilities



$p_1, p_2, \dots, p_M$ , respectively. In the gamma generator,  $M = 2N+2$ . A cumulative probability vector  $P = (p_1, p_2, \dots, p_M)$  is defined by

$$P_i = \sum_{j=1}^i p_j \quad (i=1, 2, \dots, M).$$

If a random number  $U$  uniformly distributed on  $(0, 1)$  is generated, the selection of an event can be carried out by comparing  $U$  serially with  $p_1, p_2, \dots$  and stopping when, for the first time,  $U \leq p_n$  for some  $n$ ; event  $n$  is then selected.

This method requires on the order of  $M/2$  comparisons and would be far too slow to solve the selection problem.

Substantial saving of comparisons results from applying the binary search method to the problem of finding  $n$  such that  $p_{n-1} < U \leq p_n$ . For this scheme, an interval of uncertainty is defined by selection of the indices  $i$  and  $j$  so that  $p_i \leq U \leq p_j$  at all times; initially,  $i$  is taken to be zero (with  $p_0 = 0$ ) and  $j$  to be  $M$ .  $U$  is then compared with  $p_{[(i+j)/2]}$  and the interval of uncertainty adjusted according to the result. This procedure is continued until the difference  $i-j$  is equal to 1; then  $n = j$ . It is not difficult to show that  $\log_2 M$  comparisons are needed to find  $n$ , so that this method is considerably faster than the sequential search described previously.

A further saving of comparisons is still possible, however. Since the exact distribution of the  $p_i$ 's is known,



the search procedure can be "skewed" so that more probable events can be found in fewer comparisons, thus minimizing the average number of comparisons. This can be done by comparing  $U$  with the value of  $P_k$  which is closest to the

average of  $P_i$  and  $P_j$  and proceeding as in binary search.

Finding the value of  $k$  for given  $i$  and  $j$  is no longer as simple, however; in the actual method used, two arrays of indices called Last and Next are precalculated. If  $U$  is less than  $P_k$  the following comparison is made with  $P_{\text{Last}(k)}$ ;

if  $U$  is greater than  $P_k$  then  $P_{\text{Next}(k)}$  is used for the succeeding comparison. Termination of the search is indicated when  $\text{Last}(k) = 0$  (choose event  $k$ ) or  $\text{Next}(k) = 0$  (choose event  $k+1$ ).

The binary search scheme used here is closely related to Huffman (minimum length) codes in information theory and is essentially similar to Knuth's optimum binary search tree algorithm [15]. It can be shown [15] that between  $H(P)$  and  $H(P) + 2$  comparisons are required for this method, where  $H(P)$  is Shannon's entropy function

$$H(P) = \sum_{i=1}^M p_i \log_2 (1/p_i).$$

For example, for  $k=0.1$  it was found that  $M = 2N + 2 = 164$  and that less than 6.4 comparisons were needed on the average to select an event.

## 2. Rectangle Method

If the binary search scheme selects rectangle  $i$  and if  $V$  is a second uniform  $(0,1)$  random variable independent of  $U$ , then





$$Z = x_i + Vh_i$$

is a random number selected from the  $i$ -th rectangle.

### 3. Rejection Method for Wedges

Knuth [14] and Marsaglia [23] discuss in some detail a method for sampling a random variable whose density function is almost linear. This method is used without modification to sample from the wedges.

The values of  $a_i$  and  $b_i$  were found for each wedge so that the gamma density function  $f(x)$  lies within the strip indicated in Figure 2, that is so that

$$a_i - b_i(x - x_i)/h_i \leq f(x) \leq b_i - b_i(x - x_i)/h_i$$

$$\text{for } x_i \leq x \leq x_{i+1}.$$

Since  $f(x)$  is concave upwards for all  $x$  and since the

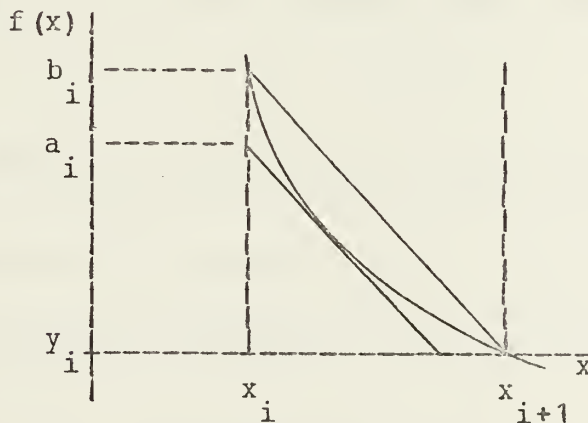


Figure 2. Sampling from the  $i$ -th wedge.

algorithm is fastest for  $a_i$  close to  $b_i$ , the lower line is taken to be the tangent to the density function curve



parallel to the chord from  $(x_{i+1}, y_i)$  to  $(x_i, k_i)$ . The tangent can be found by Newton-Raphson iteration.

The wedge algorithm is then:

(1) Generate two uniform random numbers,  $U$  and  $V$ ; if  $U > V$  then exchange  $U$  and  $V$ .

(2) Set  $Z$  to  $x_i + Uh_i$ .

(3) If  $V \leq r_i = (a_i - y_i) / (b_i - y_i)$  go to step (5).

(4) If  $V > U + f(Z)/b_i$  go back to step (1).

(5)  $Z$  is the desired variate; stop.

When  $a_i$  and  $b_i$  are close together then  $r_i$  will be nearly one and the algorithm will terminate as a result of step (3) most of the time. For most of the wedges in the gamma generator, values of  $r_i$  are greater than .85 so that  $f(x)$  needs to be evaluated (step (4)) less than 1/3 of the time.

Knuth [14] gives a proof that this method works properly.

## B. APPROXIMATIONS FOR THE INVERSE GAMMA CDF

Any continuous random variable can be sampled if its inverse distribution function  $F^{-1}$  can be found, for if  $U$  is a uniform  $(0,1)$  variate then  $Z = F^{-1}(U)$  is a variate from the desired distribution. Unfortunately, there is no simple inverse for the gamma distribution function with shape parameter less than one; Phillips and Beightler [26] attempted a rational approximation for the inverse with



little success (see Subsection III.E.2). In the present method, two different inverse approximations are used for the head and tail regions. No inverse is needed elsewhere since the wedge sampling technique described above uses  $f(\cdot)$ , the gamma density function, and not the gamma inverse.

## 1. Low Values of Z

### a. Method

The series expression for the incomplete gamma function (gamma CDF) can be expanded as follows:

$$\begin{aligned} F(x) &= g(k; x) \\ (3-2) \quad F(x) &= \frac{1}{\Gamma(k)} \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+k}}{n! (n+k)} \\ &= \frac{x^k}{\Gamma(k)} \left[ \frac{1}{k} - \frac{x}{k+1} + \frac{x^2}{2(k+2)} - \dots \right]. \end{aligned}$$

If  $Z = F(x)$  then

$$\begin{aligned} k\Gamma(k)Z &= x^k \left[ 1 - \frac{k}{k+1}x + \frac{k}{2(k+2)}x^2 - \dots \right] \\ [\Gamma(k+1)Z]^{1/k} &= x \left[ 1 - \frac{k}{k+1}x + \frac{k}{2(k+2)}x^2 - \dots \right]^{1/k}. \end{aligned}$$

When  $\frac{k}{k+1}x \ll 1$  the first-order approximation to the inverse,  $x_1$ , can be used. This corresponds to the case when the variable is to be sampled from that part of the head region close to zero.

$$\begin{aligned} x_1 &= [\Gamma(k+1)Z]^{1/k} \\ &= F^{-1}(Z) \end{aligned}$$



When the  $\frac{k}{k+1} x$  term cannot be ignored, a better approximation is obtained by solving

$$\begin{aligned} x_1 &= x \left[ 1 - \frac{k}{k+1} x + O(x^2) \right]^{1/k} \\ &\approx x \left[ 1 - \frac{1}{k} \cdot \frac{k}{k+1} x + O(x^2) \right] \\ &\approx x - \frac{1}{k+1} x^2 + O(x^3) \end{aligned}$$

for the second-order approximation,  $x_2$ . The result is

$$\begin{aligned} x_2 &= (1/2) \left[ (k+1) - \sqrt{(k+1)^2 - 4(k+1)x_1} \right] \\ &\approx F^{-1}(Z). \end{aligned}$$

A more convenient form of  $x_2$  for computation is

$$x_2 = (2x_1) / \left[ 1 + \sqrt{1 - 4x_1/(k+1)} \right].$$

#### b. Accuracy

The value of  $P_0$  (probability of the head region) is determined empirically for the gamma generator by the formula

$$P_0 = 10^{-5} / k^2$$

so that the accuracy of the approximation to the inverse incomplete gamma function was investigated for values of  $Z$  in the range zero to  $P_0$  for various values of  $k$ . It was found that the approximation improved as  $Z \rightarrow 0$  and as





$k \rightarrow 0$ ; in no case did the relative error exceed  $10^{-9}$  in double precision or  $10^{-7}$  in single precision.

## 2. Values of Z Near One

### a. Method

Newton-Raphson iteration is used for approximating the inverse gamma distribution function for values of  $Z$  near one; that is the equation

$$h(x) = g(k; x) - Z = 0$$

is solved for  $x$  by using Newton's formula

$$x_{n+1} = x_n - h(x_n) / h'(x_n).$$

Six-digit convergence can be achieved after four or five iterations of this method starting with  $x_1 = 1.0$ .

Application of the method requires evaluation of  $h(x)$  and  $h'(x)$ . The latter is just the gamma density function

$$h'(x) = f(x) = \frac{1}{\Gamma(k)} x^{k-1} e^{-x},$$

while the former can be found by summing the series (3-2). Since  $x$  is likely to be very large when  $Z$  is near one, however, as many as 30 terms of the series must be summed for convergence with a resultant loss of speed and the accumulation of serious round-off errors. For this reason,  $g(k; x)$  is evaluated by a continued fraction approximation [1] which has the added advantage of producing the value of  $1 - g(k; x)$ . Since the method is applied only to values of  $Z$



in the range (0.999,1.000) the ability to deal with 1-Z leads directly to a net gain in significant digits.

#### b. Accuracy

It was difficult to investigate the accuracy of this approximation in the same way as for the previous method because of the loss in significant digits in  $1 - Z$  as  $Z \rightarrow 1$ . A series of values of  $y$  in the range .00001 to .00100 was defined; for each value the approximation was used to find  $x = F^{-1}(1-y)$  and then  $z = F(x)$  was obtained by the continued fraction approximation. The relative error between  $z$  and  $1-y$  was in most cases zero and in no case greater than  $4 \cdot 10^{-7}$ .

### C. DESCRIPTION OF COMPUTER PROGRAM

#### 1. Program Structure

The gamma generator was implemented as a FORTRAN-callable subroutine with each call returning a single floating point (type REAL) gamma variate. One version was written in the IBM implementation of FORTRAN IV while another was written in a combination of FORTRAN and IBM 360/Assembler F. Copies of listings for both programs are on pages 77-84 and 85-88.

The basic approach in both generators is to divide the subroutine into two parts: a relatively lengthy (and slow) initialization section which calculates tables and constants and a short, fast section which is called repeatedly to handle the actual generation process. The initialization part of the program was given the name GMINIT (for Gamma INTialization) while the generator was called



GAMA. Calling sequences and arguments are discussed in the following Subsection.

GMINIT accepts values of  $k$  in the range  $0.05 \leq k < 1.00$ . It produces the cumulative vector PROB (analogous to  $P$  in Subsection III.A.1); before being placed into PROB the component probabilities are ordered with the smallest first to minimize round-off errors. GMINIT also finds the arrays NEXT and LAST for use in the binary search method as well as the arrays X, H, R and B for sampling from the rectangles and wedges. Finally, GMINIT calculates constants for the head and tail approximations. Sample output from GMINIT is on pages 70-72.

Two further subroutines were also written: IGAM(K,X) which evaluates the incomplete gamma function  $g(k;x)$  and INVGAM(K,Z) which uses Newton-Raphson iteration to solve the equation  $g(k;x) = 1-Z$  for  $x$ . INVGAM was also programmed in both FORTRAN and Assembly language. Listings for both IGAM and INVGAM are on pages 89-90 and 91-95, respectively.

In addition, a uniform random number generator is required. Since an average of about 2.5 uniform numbers are required for each gamma variate, timing characteristics of the uniform generator are a significant factor in the performance of the gamma generator. The particular generator used is called RANDOM and was developed by Lewis, Goodman and Miller [19]. RANDOM is very fast; it is capable of generating a uniform deviate in about 15 microseconds.

The generating routine, GAMA, carries out binary search with a single uniform random number and then samples from the proper sub-distribution using one or more additional independent uniform variates. It also performs scaling on the generated variate by multiplying it by a scale factor BETA which is an input parameter to GMINIT.

For small values of  $k$  (less than 0.1) there is some probability of a fixed point underflow error, i.e. that the generated variate will be less than the smallest possible



floating point number (about  $10^{-78}$  for the IBM 360). The FORTRAN standard fixup for this error is to return a result of zero, which might be satisfactory for some applications; it could create serious problems, however, if, for example, a log transformation of the gamma variates is required. If a large enough scale factor is used, modification of GAMA to perform prescaling before applying the low value approximation would reduce the problem.

## 2. Implementations

### a. FORTRAN IV Version

In the FORTRAN version of the generator the instruction sequence

```
CALL GMINIT(K,BETA)
```

```
...
```

```
CALL GAMA(Z,IX)
```

results in the initialization of the generator for shape parameter  $K$  and scale parameter  $\lambda = 1/\text{BETA}$  and the assignment of a gamma random value to  $Z$ .  $IX$  is an integer seed for the random number generator `RANDOM` and should not normally be modified in the main program. The same sequence of gamma deviates can be reproduced at any time, however, by re-initializing  $IX$  to its original value.

IBM FORTRAN IV has several features that are not in American National Standard FORTRAN and which may not be implemented in other compilers. Two of these are used in the generator and its subroutines: the built-in function `GAMMA(X)` which computes the gamma function of its argument and the `ENTRY` statement which allows a subroutine to have multiple entry points. The `ENTRY` option is used to cut down on overhead in differentiating which of the two parts (`GMINIT` or `GAMA`) of the subroutine is being called; it is





equivalent to having two independent subroutines sharing a COMMON block. Besides modification of these areas, conversion to another computer would require adjustment of convergence test values in INVGAM, IGAM and GMINIT.

#### b. Assembler Version

The Assembler version of the generator was written to save execution time in GAMA; the instruction sequence

```
CALL START(K,BETA,IX)
```

```
...
```

```
Z = GAMA(0)
```

has the same effect as the previous sequence. Linkage overhead in GAMA has been substantially reduced, however, since no arguments need be passed (the zero in the instruction above is a dummy argument and is required by FORTRAN conventions). The capability of re-initializing the seed for RANDOM was not retained in this version.

START is a dummy section of GAMA which calls on a modified version of GMINIT. Since execution time was a secondary consideration for the initialization routine (which was normally called only once), GMINIT was maintained in FORTRAN for ease of programming. The only modifications to GMINIT in this case are in the calling sequence and in the values produced for the arrays NEXT and LAST, which were changed to allow for easier search.

### 3. Timing and Core Requirements

The generator was written with the sole aim of achieving the fastest possible execution time without regard for storage requirements. The execution times obtained were highly dependent on k; for example, more rectangles are needed for lower values of k so that the binary search method occupies more time in these cases. Samples of 10,000



variates were run on the Naval Postgraduate School IBM 360/67 computer and the generation times recorded for several different values of  $k$ ; the average times per variate (in microseconds) were as follows:

$k$	FORTRAN IV Version G Compiler	H Compiler	Assembler Version	Johnk's Method
0.10	430	327	201	824
0.25	385	310	189	864
0.50	350	279	186	889
0.75	339	273	178	875
0.90	325	270	179	803
Average	366	291	187	851

By way of comparison, an implementation of Marsaglia's normal generator on the same computer requires about 60 microseconds per variate while a Marsaglia exponential generator takes 70 microseconds per variate. Johnk's method is discussed in Subsection III.E.3; it is included here for purposes of comparison.

Although this method of evaluating run times is easy to implement and gives a realistic estimate of what can be expected in practice, it is somewhat misleading since competition for system resources between jobs in a multiprogrammed computer causes considerable variability in observed run times for a given job. The observations above were recorded when the system was relatively idle; on a busy afternoon these times may increase by as much as 15 per cent.

The times to complete the set-up phase in GMINIT were also highly dependent on  $k$ ; the following run time values were observed:

$k$	Set-up Time (seconds)
0.10	.893
0.25	.348
0.50	.212
0.75	.160



Since a much larger number of rectangles is required for decomposition when  $k$  is small, it can be seen that the additional computation time for set-up in these cases can be quite costly. In fact, enough time is used to generate between 900 and 4500 variates, depending upon the value of  $k$ .

Both the Assembler and FORTRAN versions of the generator require excessive core storage, at least in comparison with other methods for generating random variates (see, for example, Ahrens and Dieter [2]). At the Naval Postgraduate School installation, where the minimum job core allotment is 100,000 bytes (25,000 words), the size of the program was not a restriction, although it could be at another installation. Both versions of the subroutine require, in addition to space for their own code and the tables, core storage for the FORTRAN built-in functions EXP, DEXP, ALOG, DLOG, GAMMA, DGAMMA and SQRT, as well as the user subroutines IGAM, INVGAM and RANDOM. Core requirements (in bytes) for the various subroutines are as follows:

	FORTRAN IV Version		Assembler
	G Compiler	H Compiler	Version
GAMA	13206	12400	5664
GMINIT			6860
INVGAM	1070	866	684
IGAM	740	574	740
RANDCM	320	320	320
Functions	5056	5056	5056
Total	20392	19216	19324

Further improvements in both space requirements and run time may still be possible through the use of in-line coding instead of subroutine calls. These modifications would have the further result of making the generator independent of vendor supplied mathematical subroutines. Ahrens and Dieter [2] point out some advantages of this approach in their improvement of Marsaglia's normal generator.



## D. STATISTICAL TESTING

Having thoroughly investigated the numerical accuracy of the various approximations used in the generator, it was then necessary to test the generated variates for "statistical accuracy". Formally, the null hypothesis

$H_0$ : The generated deviates  $z_1, z_2, \dots$  were sampled from a unit gamma distribution with shape parameter  $k$ .

was to be tested against the alternative

$H_1$ : The generated deviates were not sampled from the unit gamma distribution.

Of the wide variety of standard goodness-of-fit tests, two were chosen for testing  $H_0$ : the chi-square test and the Anderson-Darling test. In addition, the reproductive property of the gamma distribution (that is, that the sum of two gamma random variables with shape parameters  $k_1$  and  $k_2$  and equal scale parameters is also a gamma random variable with shape parameter  $k_1 + k_2$ ) was exploited in a test on the sums of several variates.

### 1. Chi-square Test

A modification of the chi-square test suggested by Naylor [25] for uniform random number generators was carried out. For this test, a set of  $n$  quantiles of the object distribution (i.e., the gamma distribution) is first found;





the  $i$ -th quantile  $r_i$  is the solution to the equation

$$F(r_i) = i / n .$$

Quantiles for the gamma distribution are unique. A sample of size  $N$  variates is then generated and the number of variates  $f_i$  falling into each interval  $r_{i-1} \leq z \leq r_i$  is recorded. The statistic

$$X_1^2 = \frac{n}{N} \sum_{i=1}^n (f_i - \frac{N}{n})^2$$

is then found for the sample;  $X_1^2$  has approximately a chi-square distribution with  $n-1$  degrees of freedom. This procedure is then repeated  $M$  times and a second statistic  $X_2^2$  is calculated in an analogous manner, using  $m$  quantiles of the chi-square distribution with  $n-1$  degrees of freedom. The value of  $X_2^2$  is then used to accept or reject  $H_0$ .

Values chosen for the actual test were  $m = n = 10$ ,  $N = 1000$  and  $M = 100$ . Thus, each test involved generation of 100,000 variates. Tabulated results for various values of  $k$  are as follows:

$k$	$X_2^2$	Chi-square Level	Minimum $X_1^2$	Maximum $X_1^2$
.06	4.8	.149	2.24	27.28
.10	4.6	.132	1.80	22.20
.15	11.2	.738	1.90	23.74
.50	4.4	.117	1.66	21.58
.90	11.4	.751	1.66	25.68
.95	8.8	.544	1.90	21.98



The column headed "Chi-square Level" gives the quantile of the chi-square distribution with nine degrees of freedom at the observed value of  $\chi^2_2$ . Based on these results,  $H_0$  could not be rejected.

There are some serious drawbacks to the chi-square test, not the least of which is the fact that the statistic depends strongly on the specific quantiles taken and is not an invariant for a particular sample. Nevertheless, the test proved to be quite sensitive even to minor variations in the generator. For example, it readily detected the statistical effect of the omission of a single wedge (whose probability was .002) by the binary search procedure, producing composite scores greater than 50 in this case. As is indicated in the sequel (Subsection III.E.2), the test was also used to reject the Phillips generator, which achieved scores greater than 800 for small values of  $k$ .

## 2. Anderson-Darling Test

The test of goodness-of-fit proposed by Anderson and Darling [3] is a distribution-free test in that instead of the observed values of the variate  $x_1, x_2, \dots, x_n$  the values

$$u_i = F(x_i)$$

are used. If  $F(x)$  is in fact the distribution function of the  $x_i$ 's (that is, if the null hypothesis is true), then the  $u_i$ 's are a sample of size  $n$  from a uniform  $(0,1)$  distribution. If the  $u_i$ 's are now ordered so that  $u_{(1)} \leq$



$u_{(2)} \leq \dots \leq u_{(n)}$ , then the Anderson-Darling statistic

$$W_n^2 = -n - \frac{1}{n} \sum_{i=1}^n [ (2i-1) \ln u_{(i)} + \{2(n-i) + 1\} \ln (1 - u_{(i)}) ]$$

can be computed. The null hypothesis can then be accepted if  $W_n^2$  is not too large. Lewis [17] has determined the distribution of  $W_n^2$  for various values of  $n$ .

It should be mentioned that the Anderson-Darling statistic gives the most weight to observations from the tails of the distribution, where  $u_{(i)}$  is very close to zero or one. Thus, the test was ideal for the gamma generator, since the rectangle-wedge technique had been successfully applied in a wide number of other cases and the main uncertainty for the generator lay in the statistical properties of the head and tail approximations.

Accordingly, Anderson-Darling statistics were found for 100 samples of 100 variates each. The results of Lewis [17] were then used to find ten quantiles of the distribution of  $W_{100}^2$  so that a chi-square statistic could be computed for the sample of 100 Anderson-Darling statistics. The results were as follows:

k	Chi-square Statistic	Chi-square Level	Minimum $W_n^2$	Maximum $W_n^2$
.10	4.269	.107	.178	3.500
.50	4.895	.157	.175	5.071
.90	4.306	.110	.223	5.582



Once again it was decided to accept  $H_0$  on the basis of the test.

### 3. "Reproductive" Property Test

It is well-known (Cramer [8]) that the sum of two independent gamma random variables with identical scale parameters and respective shape parameters  $k_1$  and  $k_2$  also has a gamma distribution with shape parameter  $k_1 + k_2$ . Since a gamma random variable with  $k=1.0$  is an exponential variable, a possible test for  $H_0$  is to investigate whether the statistic

$$V_n = \sum_{i=1}^n z_i,$$

has a unit exponential distribution, where the  $z_i$ 's are generated by GAMA with  $k = 1/n$ .

The test was carried out for samples of 100  $V_n$ 's with  $n = 2, 4, 6, 8, 10$ . For each of four such samples, an Anderson-Darling statistic was computed and the maximum value of the statistic recorded. The approximate significance level of the statistic was determined by linear interpolation in the table of the asymptotic distribution of  $W_n^2$  given by Lewis [17]. The results are as follows:





n	k	Maximum $W_{100}^2$	Approximate Significance Level
2	.500	1.577	.841
4	.250	2.045	.913
6	.167	1.031	.658
8	.125	1.469	.816
10	.100	1.525	.829

As a further test of whether the statistics  $V_n$  have an exponential distribution, the statistic

$$V'_n = \text{Minimum}(V_{n;1}, V_{n;2}, \dots, V_{n;1000})$$

was investigated; this was done specifically to test the statistical effect of the low value approximation to  $F^{-1}(Z)$ .

It is well-known that the minimum of  $m$  unit exponential random variables is an exponential random variable with mean

$1/m$ . Thus, samples of 100 observations of  $V'_n$  were generated

for  $n=2, 4, 8$  and an Anderson-Darling statistic computed using the exponential distribution function with mean .001. The results are as follows:

n	k	$W_{100}^2$	Approximate Significance Level
2	0.500	.903	.588
4	0.250	.431	.182
8	0.125	2.287	.936

As can be seen,  $H_0$  is readily accepted based on these tests.



## E. COMPARISON WITH OTHER METHODS

### 1. Naylor's Method

The method of Naylor [25] for producing gamma variates with non-integral shape parameter was not seriously considered for the simulation since it is not defined for  $k < 1.0$ . The method involves sampling a mixture of two gamma random variables with integral shape parameters  $\lfloor k \rfloor$  and  $\lfloor k \rfloor + 1$  (the notation  $\lfloor k \rfloor$  denotes the truncated integer part of  $k$ ). Berman has shown [28] that the method is satisfactory for small simulations when  $k \geq 5$  but his results demonstrate that the method is not statistically acceptable for sample sizes on the order of 100,000 for any value of  $k$ .

The timing results for the Naylor method obtained by Berman [28] for an IBM 360/65 computer suggest that a better way to generate non-integral gamma random numbers is to add an integral gamma variate with shape parameter  $\lfloor k \rfloor$  and a variate from GAMA with shape parameter  $k - \lfloor k \rfloor$ . For  $k > 10$  this modification adds less than 20 per cent to the execution time but renders the result statistically exact.

### 2. Phillips's Method

Phillips and Beightler [26] proposed the use of a rational approximation to the inverse gamma distribution function for generating gamma variates; three different approximations were developed for  $0 < k \leq 2.0$ ,  $2.0 < k \leq 5.0$  and  $k > 5.0$ . The technique was extensively tested by Phillips using the Kolmogorov-Smirnov goodness of fit test; unfortunately, this test is notoriously insensitive to deviations in the tails of the distribution. The chi-square goodness of fit test was therefore applied to the method for



$k < 1.0$  and it was found that the hypothesis that the Phillips numbers were sampled from the gamma distribution could be rejected at any desired confidence level (chi-square scores in excess of 800 on 9 degrees of freedom were observed). The method was not investigated for  $k > 1.0$ . Furthermore, this method required substantially more execution time than the method developed here (on the order of 600 microseconds per variate).

### 3. Johnk's Method

Johnk's method [13] was the only known theoretically exact means of generating gamma variates with non-integral shape parameter. It is a rejection technique, like the wedge method of Subsection III.A.3. For generating gamma variates with shape parameter  $k$  less than 1.0 the method can be set forth as follows:

(1) Generate two independent uniform (0,1) random variates  $U$  and  $V$ .

(2) Set  $X = U^{1/k}$  and  $Y = V^{1/(1-k)}$ .

(3) If  $X + Y > 1.0$  go back to (1).

(4) Generate a new uniform variate  $R$  and set  $W = \ln R$ .

(5)  $Z = W \cdot X / (X+Y)$  is a unit gamma variate. Scale  $Z$  if required and stop.

Translations of Johnk's original proofs of the correctness of this method are contained in the Phillips and Beightler [26] and Fox [9] papers. Essentially the method generates a beta variate and then uses the Lukacs-Laha results [16,21] to produce a gamma variate. Johnk shows that when  $k$  is less than 1.0 the rejection in step (3) will occur between 1.00 and 1.27 times per variate on the average.

Johnk's method is not satisfactory for large-scale simulation use because it requires excessive time; see



Subsection III.C.3 for the observed times for a FORTRAN implementation of the method. The time can be improved by recognizing that if  $U$  is a uniform  $(0,1)$  variate then  $-\ln U$  is a unit exponential variate. The time to generate  $U$  and evaluate its natural logarithm is about 130 microseconds on the IBM 360/67 computer while an assembler implementation of Marsaglia's exponential algorithm can deliver an exponential variate in about 70 microseconds. The improved algorithm is then

- (1) Generate two independent exponential variates,  $R$  and  $S$ .
- (2) Set  $X = e^{-R/k}$  and  $Y = e^{-S/(1-k)}$ .
- (3) If  $X + Y > 1$  go back to (1).
- (4) Generate a new exponential variate  $T$ .
- (5) Deliver  $Z = T \cdot X / (X+Y)$ . Stop.

Even with these improvements, however, the method still requires over 500 microseconds per variate in an assembler version. Thus, allowing for GAMA's substantial set-up time, it is faster to use the Johnk method for less than 250 variates ( $k=0.75$ ) to 1,350 variates ( $k=0.1$ ), depending on  $k$ , but the method is too slow for large-scale simulation (100,000 or more variates). The method remains very attractive for use when time is not so critical, however, since it is very easy to program and requires minimal storage.





#### IV. COMPUTER SIMULATION EXPERIMENT

##### A. DESCRIPTION OF METHOD

The basic purpose of the simulation experiment was to investigate the distribution of  $Y_J$  for small values of  $J$ . Different simulations were run for shape parameters  $k$  of .1, .25, .5 and .75 corresponding to coefficients of variation  $C^2(X)$  of 10, 4, 2 and 1.33. In addition, a simulation using an implementation of the Marsaglia exponential generator was run for the case  $k=1.0$ .

##### 1. Simulation

Ten thousand samples of  $J$  gamma variates each were generated for  $J = 10, 30, 50, 100$  and the  $Y_J$  statistic computed for every sample. The first four sample moments were found and then the entire sample of 10,000  $Y_J$ 's was ordered in order to estimate the quantiles of the distribution (see Subsection IV.B.2). This procedure produced 27 statistics: four sample moment estimates and 23 quantile estimates. The entire procedure was then repeated ten times and the mean and variance of each of the 27 estimates found over the ten samples of each.

Obviously a great many gamma variates were required to carry out the simulation, nearly 20 million for each value of  $k$ . The size of the simulation graphically demonstrates the need for speed in the gamma generator.



Even with the relatively fast performance of the Assembly-coded generator, the total IBM 360/67 central processor time needed to run the simulation was about 60 minutes for each k value.

## 2. FORTTRAN IV Program

The simulation itself was written in FORTRAN IV; a program listing is included on pages 96-98. The program and subroutines required nearly 89,000 bytes of core at run time, 40,000 bytes of which were needed for the 10,000-element array used for quantile estimation. A sample of output from the program for J=30 is included on pages 73-76.

## B. ESTIMATION OF PARAMETERS

### 1. Distribution Moments

If  $Y_{J;i}$  represents the i-th observation of  $Y_J$  in the simulation, then the r-th moment  $m_r$  of  $Y_J$  can be estimated by

$$\bar{m}_r = \frac{1}{N} \sum_{i=1}^N (Y_{J;i})^r.$$

An unbiased estimator of the variance is then

$$\bar{s}^2 = \frac{N}{N-1} (\bar{m}_2 - \bar{m}_1^2).$$

In the simulation, estimates of the first four moments ( $r=1,2,3,4$ ) were obtained. In addition, the sample variance and the coefficients of skewness and kurtosis were



calculated from the moment estimates.

## 2. Quantiles

### a. Quantile Estimation Considerations

A quantile  $x_a$  of a distribution  $F(x)$  is defined as the solution to the equation

$$F(x_a) = a \quad (0 < a < 1).$$

Quantiles are unique for most continuous distributions but are not, in general, for discrete distributions. In the simulation, quantiles of  $Y_J$  were estimated for  $a = .001, .002, .005, .010, .020, .025, .050, .1(.1).9, .950, .975, .980, .990, .995, .998$  and  $.999$ .

There are two principal methods for quantile estimation in simulations: the order statistic estimator and the Robbins-Monro stochastic approximation technique. (For a more complete discussion, see Goodman, Lewis and Robbins [11].) The order statistic method was chosen for this simulation because of the ease of implementation. Let the  $N$  observations  $Y_{J;1}, Y_{J;2}, \dots, Y_{J;N}$  of  $Y_J$  be ordered so that  $Y_{(J;1)} \leq Y_{(J;2)} \leq \dots \leq Y_{(J;N)}$ ; then  $Y_{(J;i)}$  is called the  $i$ -th order statistic of the sample and an estimate of the  $a$ -quantile  $y_a$  of the distribution of  $Y_J$  is

$$\bar{y}_a = Y_{(J; \lfloor aN \rfloor)}.$$



## b. Sorting Random Data

In order to estimate the extreme quantiles of the distribution of  $Y_J$  it is necessary to deal with large samples of  $Y_J$ . Considerations of bias (see the next Subsection) also call for large samples. As mentioned previously, a sample size of 10,000 was chosen for quantile estimation.

Sorting such a large sample can introduce considerable overhead into the simulation, however, unless the sorting method is carefully chosen. There is a bewildering variety of known sorting methods for various purposes (Knuth [15] devotes an entire text to the subject). The particular technique selected for the simulation is due to Singleton [27]; a FORTRAN implementation of the method is capable of ordering a sample of 10,000 normalized type REAL numbers in less than four seconds. In the simulation, therefore, less than five per cent of the execution time was spent in sorting the samples.

## 3. Errors and Bias

The parameter estimates may not correspond to the true parameter values for three reasons: statistical variability, computation errors and estimator bias. Statistical variability is an inescapable reflection of the fact that a simulation is not a deterministic process, as is the summing of an infinite series. Thus, the probability that the precise expected value of an estimator will be observed is very low and may be zero for a continuous distribution. The law of large numbers, however, guarantees that the estimator will converge closer to its theoretical value as more replications of the simulation are performed.

Computation error is a reflection of digital





computer round-off errors; that is, for every computer system there is a floating-point constant  $e$  such that when  $|d| \leq |e|$

$$1.0 + d = 1.0.$$

For the IBM 360 system,  $e = 9.53 \cdot 10^{-7}$ . Thus, if very many data elements are to be summed (as in determining the moment estimators) there may be some loss of significance as the accumulated sum increases. The problem will be even more acute if very skewed data are to be added, as in the determination of  $Y_{1J}$  and  $Y_{2J}$ .

Round-off errors in  $Y_{1J}$  and  $Y_{2J}$ , however, tend to cancel out when  $Y_J$  is calculated. On the other hand, in the simulation the observed moment estimators  $\bar{m}_r$  were consistently below their theoretical values, thus reflecting considerable round-off error. In fact,  $\bar{m}_r$  was less than  $m_r$  in 17 observations out of 20 for  $r=1$  and in 14 of 20 for  $r=2$ . Computing the moment estimators using double precision arithmetic (thus decreasing the value of  $e$  to  $2.22 \cdot 10^{-16}$  and therefore greatly reducing round-off errors) resulted in values of  $\bar{m}_r$  more in agreement with theory.

Estimator bias is a theoretical property of the specific estimators used in a simulation. In this case, the  $\bar{m}_r$  are unbiased estimators of the  $m_r$  but

$$E[\bar{Y}_a] = y_a + O\left(\frac{1}{N}\right),$$



where  $N$  is the sample size (Goodman, Lewis and Robbins [11]). Thus  $\bar{y}_a$  is a biased estimator. No attempt was made to evaluate the extent of the bias of  $\bar{y}_a$ , but it is clear that the simulation results may not be accurate for more than two or three decimal places.

## C. SIMULATION RESULTS

### 1. Agreement with Theory

For each simulation run, a  $t$ -statistic was computed using the sample mean and variance. Since the sample mean is the sum of 100,000 identically distributed random variables (the observations of  $Y_J$ ), it is safe to assume that its distribution is very nearly normal and that

$$t = (\bar{m}_1 - E[Y_J]) / (\bar{s} / \sqrt{n})$$

where

$$\bar{s}^2 = \frac{n}{n-1} (\bar{m}_2 - \bar{m}_1^2)$$

has a  $t$  distribution with  $n$  degrees of freedom. When  $n$  is large (100,000 in this case) then the  $t$  distribution can be approximated by the standard normal distribution. The  $t$  statistics for the various runs are summarized in the following table:



J	10	30	50	100
k				
0.10	-0.0688	-2.2718	-0.5641	-0.9348
0.25	1.0851	-2.7454	-2.4309	-1.9453
0.50	0.1989	-3.6603	-2.5376	1.2218
0.75	-0.1440	-1.8950	-2.6556	-1.2765
1.00	-0.5925	-3.4506	-1.2623	NA

Since the .01 quantile of the standard normal distribution is -2.326, it can be readily seen that the above results do not agree with theory.

As mentioned above, the reason for this discrepancy lies in accumulating round-off errors in calculating the moment estimators. The simulations for J=30 were thus rerun with the moment estimators computed in double precision. No change in the quantile estimates was observed, but the new t statistics were

k	0.10	0.25	0.50	0.75
t statistic	-0.9206	-0.8101	-1.0592	1.3176

Thus, it is concluded that the simulation results agree with theory as far as the observed moments are concerned.

## 2. Tabulation of the Distribution of $Y_J$

In addition to the actual values for  $Y_J$ , the simulation program produced normalized values for the quantile estimates

$$\bar{n}_a = (\bar{Y}_a - u) / \sigma ,$$

where



$$u = E[Y_J]$$

$$= \frac{J+1}{2}$$

and

$$\sigma^2 = \text{Var}[Y_J]$$

$$= \frac{J-1}{12} \frac{J+1}{KJ+1}.$$

The normalized statistics are tabulated here in order to show the asymptotic convergence of the distribution of  $Y_J$  to a normal distribution; quantiles of the standard normal distribution are also listed.





<u>Alpha</u>	<u>J=10</u>	<u>J=30</u>	<u>J=50</u>	<u>J=100</u>	Normal Quantile
.001	-2.2027	-2.7519	-2.8819	-2.9941	-3.090
.002	-2.1920	-2.6114	-2.7360	-2.7978	-2.878
.005	-2.1473	-2.4244	-2.4940	-2.5415	-2.576
.010	-2.0788	-2.2439	-2.2911	-2.3166	-2.326
.020	-1.9445	-2.0244	-2.0507	-2.0549	-2.054
.025	-1.8738	-1.9473	-1.9655	-1.9672	-1.960
.050	-1.6546	-1.6654	-1.6675	-1.6547	-1.645
.100	-1.3444	-1.3222	-1.3107	-1.2983	-1.282
.200	-0.9230	-0.8844	-0.8697	-0.8563	-0.842
.300	-0.5885	-0.5541	-0.5441	-0.5364	-0.524
.400	-0.2773	-0.2746	-0.2608	-0.2613	-0.253
.500	-0.0012	-0.0051	0.0023	-0.0029	0
.600	0.2802	0.2687	0.2685	0.2604	0.253
.700	0.5923	0.5535	0.5535	0.5382	0.524
.800	0.9257	0.8783	0.8732	0.8566	0.842
.900	1.3426	1.3177	1.3042	1.2958	1.282
.950	1.6541	1.6553	1.6621	1.6577	1.645
.975	1.8754	1.9228	1.9537	1.9627	1.960
.980	1.9437	2.0036	2.0467	2.0523	2.054
.990	2.0777	2.2178	2.2859	2.3096	2.326
.995	2.1498	2.4068	2.5013	2.5484	2.576
.998	2.1901	2.6027	2.7520	2.8258	2.878
.999	2.2009	2.7274	2.9476	3.0085	3.090
u	5.5	15.5	25.5	50.5	
$\sigma$	2.0310	4.3277	5.8914	8.7034	
t-Test for Mean	-0.0688	-2.2718	-0.5641	-0.9348	

Table I. Simulation results for  $k=0.10$ . Normalized quantiles of the distribution of  $Y_J$  under the null hypothesis  $b=0$  (no trend) are tabulated.



<u>Alpha</u>	<u>J=10</u>	<u>J=30</u>	<u>J=50</u>	<u>J=100</u>	Normal Quantile
.001	-2.6635	-3.0231	-2.9940	-3.0984	-3.090
.002	-2.5614	-2.7949	-2.8016	-2.8912	-2.878
.005	-2.3792	-2.5528	-2.5519	-2.5871	-2.576
.010	-2.2094	-2.3081	-2.3277	-2.3405	-2.326
.020	-2.0089	-2.0596	-2.0612	-2.0672	-2.054
.025	-1.9340	-1.9722	-1.9727	-1.9775	-1.960
.050	-1.6581	-1.6723	-1.6639	-1.6715	-1.645
.100	-1.3211	-1.3174	-1.2980	-1.2982	-1.282
.200	-0.8811	-0.8663	-0.8604	-0.8583	-0.842
.300	-0.5511	-0.5414	-0.5375	-0.5339	-0.524
.400	-0.2643	-0.2608	-0.2632	-0.2585	-0.253
.500	0.0089	-0.0016	-0.0070	0.0015	0.
.600	0.2673	0.2573	0.2568	0.2573	0.253
.700	0.5619	0.5365	0.5311	0.5351	0.524
.800	0.8913	0.8639	0.8589	0.8529	0.842
.900	1.3230	1.3024	1.3015	1.2946	1.282
.950	1.6566	1.6536	1.6593	1.6576	1.645
.975	1.9359	1.9642	1.9692	1.9776	1.960
.980	2.0101	2.0579	2.0545	2.0682	2.054
.990	2.2225	2.2991	2.3178	2.3388	2.326
.995	2.3901	2.5438	2.5635	2.5759	2.576
.998	2.5484	2.7867	2.8170	2.8870	2.878
.999	2.6496	2.9408	3.0409	3.0975	3.090
u	5.5	15.5	25.5	50.5	
$\sigma$	1.5353	2.9688	3.9276	5.6611	
t-Test for Mean	1.0851	-2.7454	-2.4309	-1.9453	

Table II. Simulation results for  $k=0.25$ . Normalized quantiles of the distribution of  $Y_J$  under the null hypothesis  $b=0$  (no trend) are tabulated.



<u>Alpha</u>	<u>J=10</u>	<u>J=30</u>	<u>J=50</u>	<u>J=100</u>	Normal Quantile
.001	-2.8514	-3.0592	-3.0958	-3.0890	-3.090
.002	-2.7088	-2.8700	-2.9104	-2.9028	-2.878
.005	-2.4928	-2.5669	-2.6023	-2.5900	-2.576
.010	-2.2861	-2.3410	-2.3446	-2.3538	-2.326
.020	-2.0416	-2.0851	-2.0823	-2.0925	-2.054
.025	-1.9578	-1.9918	-1.9941	-1.9955	-1.960
.050	-1.6617	-1.6761	-1.6750	-1.6605	-1.645
.100	-1.3028	-1.3107	-1.3119	-1.2956	-1.282
.200	-0.8685	-0.8723	-0.8620	-0.8468	-0.842
.300	-0.5420	-0.5460	-0.5375	-0.5201	-0.524
.400	-0.2661	-0.2660	-0.2587	-0.2471	-0.253
.500	0.0030	-0.0051	0.0022	0.0092	0.
.600	0.2665	0.2592	0.2596	0.2708	0.253
.700	0.5483	0.5375	0.5380	0.5430	0.524
.800	0.8727	0.8610	0.8622	0.8679	0.842
.900	1.3098	1.3037	1.3123	1.3172	1.282
.950	1.6615	1.6712	1.6747	1.6861	1.645
.975	1.9576	1.9847	1.9851	2.0026	1.960
.980	2.0483	2.0753	2.0827	2.0992	2.054
.990	2.2750	2.3388	2.3416	2.3790	2.326
.995	2.4723	2.5856	2.5955	2.6282	2.576
.998	2.6926	2.8816	2.8901	2.9099	2.878
.999	2.8498	3.0650	3.0638	3.1490	3.090
u	5.5	15.5	25.5	50.5	
$\sigma$	1.1726	2.1639	2.8301	4.0421	
t-Test for Mean	0.1989	-3.6603	-2.5376	1.2218	

Table III. Simulation results for  $k=0.50$ . Normalized quantiles of the distribution of  $Y_J$  under the null hypothesis  $b=0$  (no trend) are tabulated.



<u>Alpha</u>	<u>J=10</u>	<u>J=30</u>	<u>J=50</u>	<u>J=100</u>	Normal Quantile
.001	-2.9330	-3.1428	-3.1428	-3.1331	-3.090
.002	-2.7684	-2.9370	-2.9344	-2.9034	-2.878
.005	-2.5103	-2.6022	-2.5863	-2.6140	-2.576
.010	-2.2857	-2.3462	-2.3430	-2.3620	-2.326
.020	-2.0410	-2.0911	-2.0684	-2.0876	-2.054
.025	-1.9544	-1.9931	-1.9815	-1.9907	-1.960
.050	-1.6617	-1.6686	-1.6722	-1.6678	-1.645
.100	-1.3090	-1.3054	-1.3063	-1.3039	-1.282
.200	-0.8670	-0.8556	-0.8629	-0.8497	-0.842
.300	-0.5429	-0.5275	-0.5351	-0.5279	-0.524
.400	-0.2628	-0.2496	-0.2578	-0.2495	-0.253
.500	0.0024	0.0079	0.0003	0.0064	0
.600	0.2618	0.2658	0.2603	0.2631	0.253
.700	0.5410	0.5380	0.5358	0.5390	0.524
.800	0.8707	0.8640	0.8574	0.8584	0.842
.900	1.3153	1.3052	1.3056	1.3000	1.282
.950	1.6657	1.6771	1.6803	1.6708	1.645
.675	1.9598	1.9889	1.9920	1.9943	1.960
.980	2.0509	2.0705	2.0849	2.0894	2.054
.990	2.2908	2.3475	2.3639	2.3518	2.326
.995	2.5102	2.6129	2.6022	2.5834	2.576
.998	2.7933	2.8948	2.9219	2.8904	2.878
.999	2.9584	3.1079	3.0848	3.1105	3.090
u	5.5	15.5	25.5	50.5	
$\sigma$	0.9852	1.7855	2.3257	3.3112	
t-Test for Mean	-0.1440	-1.8950	-2.6556	-1.2765	

Table IV. Simulation results for  $k=0.75$ . Normalized quantiles of the distribution of  $Y_J$  under the null hypothesis  $b=0$  (no trend) are tabulated.





<u>Alpha</u>	<u>J=10</u>	<u>J=30</u>	<u>J=50</u>	<u>J=100</u>	Normal Quantile
.001	-2.9858	-3.0964	-3.0550	NA	-3.090
.002	-2.7774	-2.8721	-2.8522		-2.878
.005	-2.5046	-2.5750	-2.5525		-2.576
.010	-2.2994	-2.3304	-2.3006		-2.326
.020	-2.0442	-2.0496	-2.0350		-2.054
.025	-1.9502	-1.9566	-1.9455		-1.960
.050	-1.6517	-1.6464	-1.6489		-1.645
.100	-1.2917	-1.2846	-1.2775		-1.282
.200	-0.8543	-0.8440	-0.8348		-0.842
.300	-0.5284	-0.5244	-0.5172		-0.524
.400	-0.2540	-0.2533	-0.2409		-0.253
.500	-0.0009	-0.0004	0.0075		0
.600	0.2562	0.2566	0.2607		0.253
.700	0.5286	0.5238	0.5317		0.524
.800	0.8519	0.8422	0.8443		0.842
.900	1.2905	1.2875	1.2843		1.282
.950	1.6536	1.6417	1.6508		1.645
.975	1.9521	1.9635	1.9690		1.960
.980	2.0417	2.0588	2.0541		2.054
.990	2.2855	2.3152	2.3165		2.326
.995	2.4933	2.5520	2.5828		2.576
.998	2.7571	2.8427	2.8756		2.878
.999	2.9209	3.0438	3.0758		3.090
u	5.5	15.5	25.5		
$\sigma$	0.8660	1.5546	2.0207		
t-Test for Mean	-0.5925	-3.4506	-1.2623		

Table V. Simulation results for  $k=1.00$ . Normalized quantiles of the distribution of  $Y_J$  under the null hypothesis  $b=0$  (no trend) are tabulated.



#### D. LARGE-SCALE SIMULATION TOOLS

In programming the simulation it became clear that a set of standard data-handling subroutines would have been most helpful for parameter estimation. The facilities of standard simulation languages (e.g., GPSS or CSMP) and data reduction programs (SNAP/IEDA or SPSS) are not suited to the amount or type of information generated in large-scale statistical simulations such as this one. Extensive research in the literature was thus required to find high-speed techniques for sorting, quantile estimation and statistical testing and considerable effort was expended in programming the techniques.

The sophistication of state-of-the-art methods in random number generation, for example, can result in substantial savings of computer time and core requirements but only at the expense of program complexity. Since the specific details of the method involved are usually of peripheral importance to the simulation itself, removal of the programming burden will result in more resources becoming available for the main purpose of the simulation.

Following Lewis' discussion [24] of a large-scale computer-aided statistical package, the following routines would have been of value in the  $Y_J$  simulation:

- (1) A fully-documented quantile estimation subroutine using the maximum transformation Robbins-Monro stochastic approximation technique.

- (2) One or more efficient goodness-of-fit test subroutines: t-test, chi-square test, Anderson-Darling test, Kolmogorov-Smirnov test.

- (3) A sorting routine equal in speed to ACM Algorithm 347 [27].



(4) An efficient subroutine for accumulating moment estimators with minimal round-off error.



## V. CONCLUSION

### A. APPLICABILITY OF RESULTS

#### 1. Gamma Generator

In view of the wide variety of physical events which can be successfully modeled empirically as gamma renewal processes, the availability of an exact, high-speed gamma variate generator should enable more extensive simulations to be conducted in many applications areas. If the shape parameter remains constant throughout a given application, pre-calculation of GAMA's tables and constants will save set-up time in succeeding runs; if not, then the trade-off between GAMA and the Johnk technique (Subsection III.E.3) should be considered in selecting a generator. Nevertheless, it is clear that whenever a large number of gamma variates with shape parameter less than one is required, GAMA is superior to any other known method for generating them.

#### 2. Tests for Trend

The results of Section II.D indicate that the variance of the standard B test for the Poisson process needs to be "inflated" by the coefficient of variation squared for use in a gamma renewal process. Thus, it may now be possible to accept the null hypothesis that a given process is without trend when prior to this result it was not. For example, Lewis and Shedler [20] in analyzing computer page exception data obtained B values which were





far too high to accept the null hypothesis (normalized values of -2.83, -8.67 and -18.11 were observed; the .01 quantile is -2.33). However, allowing for the extremely high coefficients of variation involved (3.34, 3.27 and 3.70), the null hypothesis can be accepted in two of the three cases.

Since in practice the actual value of  $k$  is seldom known, the results obtained in Chapter II suggest that the estimated sample coefficient of variation could be used as a multiplier of the Poisson variance in applying the  $B$  test. The simulation results indicate that this procedure may lead to slightly higher acceptance levels in the gamma case when  $k$  is small, but the effect diminishes with increasing  $J$ .

Without investigation of the relative power of the  $Y_J$  test against other standard trend tests it may not be wise to apply it to an arbitrary process. It would appear most useful, in any case, when small values of  $k$  are involved, for this is precisely the situation which the  $B$  test does not handle well.

As for the tabular results, Tables I - V are sufficiently accurate for routine application of the test based on  $B$ , especially since the level of the test is arbitrary. Possible inaccuracies in the tabular values might be significant, though, if the values were used to investigate the power of the  $Y_J$  test against the alternatives  $b \neq 0$ . This has not been done.

The distributions of  $Y_J$  are in all cases observed to be approximately symmetric and underdispersed with respect to the normal distribution. Convergence to the normal distribution is rapid, the rate being slowest, as expected, for  $k=0.1$ . Even there, the normal approximation is adequate for  $J=50$  if a level of no higher than 0.990 and no lower than 0.010 is required. The approximate rates of



convergence of  $Y_J$  are such that it appears possible to use the normal approximation whenever  $J$  is greater than the following values:

k	0.10	0.25	0.50	0.75	1.00
J	100	50	30	30	10

## B. AREAS FOR FURTHER STUDY

### 1. Gamma Generator

The problem of a general purpose gamma generator ( $k \geq 1$  as well as  $k < 1$ ) should be investigated in order to extend the present work to wider applications areas. A straightforward way to do this is to follow Berman's method [28] of adding  $\lfloor k \rfloor$  exponential variates to a gamma variate with shape parameter  $k - \lfloor k \rfloor$ . This approach has two distinct drawbacks in the present case:

(1) When  $k - \lfloor k \rfloor$  is small, GAMA requires much more setup and generation time than when  $0.5 \leq k - \lfloor k \rfloor < 1.0$ ; when  $k - \lfloor k \rfloor < 0.05$ , GAMA cannot be used at all. This difficulty could be overcome by using the fact that the square of a standard normal random variable has the gamma distribution with  $k=0.5$ ,  $\lambda=0.5$ ; thus, if  $Z$  is a standard normal variate then  $Y = 2Z^2$  is a unit gamma variate with shape parameter 0.5. If a Marsaglia normal generator is available,  $Y$  can be readily generated and added to a series of exponential variates to allow GAMA to be called with shape parameter in the optimum range. A thorough investigation of the time requirements for the various options is needed to find the fastest method in any given case.



(2) When  $k$  is large, generation of ' $k$ ' exponential variates may require excessive time. In this case, taking the logarithm of the product of ' $k$ ' uniform variates may be a more efficient way to proceed.

It may be possible to overcome both these difficulties by extending the decomposition technique to values of  $k$  greater than one. In this case, however, the density function is no longer monotone and automatic generation of the sub-distributions and their probabilities will be considerably more difficult.

## 2. Tests for Trend

As mentioned previously, a more thorough investigation of the theoretical properties of the  $Y_J$  test is also indicated. Its relative power compared to other tests against the trend model should be determined; possible examples of such tests include a normal theory regression test for trend after a logarithmic transformation or the various non-parametric tests for trend that are essentially equivalent to Kendall's rank correlation test. Results should also be obtained for other types of renewal process.



GENERATED VALUES FOR K= 5.000000E-01 BETA= 1.000000E 00

HEAD PROBABILITY= 2.000011E-04 TAIL PROBABILITY= 4.005432E-05

RECTANGLE/WEDGE VALUES

I	X(I)	P(I)	Q(I)	R(I)	B(I)
1	1.41626E-08	2.36078E-05	1.246153E-06	9.581481E-01	1.69017E
2	3.927033E-08	3.779664E-05	1.3171779E-06	9.336984E-01	2.18192553E
3	5.497846E-08	6.030221E-05	6.785616E-06	9.154221E-01	3.3620455E
4	8.639472E-08	9.176671E-05	1.2446110E-05	9.1579554E-01	3.8722624E
5	1.7489272E-07	1.3524725E-04	2.0494345E-05	8.8609544E-01	1.40339967E
6	2.262222E-07	2.7955863E-04	4.456479E-05	8.7895645E-01	1.4316690E
7	1.028881E-06	3.976775E-04	6.6431197E-05	8.750603E-01	1.500715E
8	2.034201E-06	5.6408284E-04	7.1197931E-05	8.730119E-01	3.558142E
9	4.044841E-06	4.988284E-04	5.364939E-05	8.719643E-01	3.808430E
10	8.066121E-06	1.1305371E-03	1.9335721E-04	8.711559E-01	6.911670E
11	1.610868E-05	1.599371E-03	2.741409E-04	8.710284E-01	1.3526520E
12	3.219380E-05	2.262193E-03	3.890710E-04	8.7096354E-01	3.5676138E
13	6.438704E-05	3.199311E-03	5.4890772E-04	8.709244E-01	6.7644456E
14	1.287045E-04	4.524134E-03	7.664732E-04	8.7093229E-01	7.8827947E
15	2.573854E-04	6.396592E-03	1.0984139E-03	8.709592E-01	3.3913394E
16	5.147473E-04	9.047385E-03	1.554139E-03	8.709592E-01	3.690067E
17	1.029471E-03	1.8027759E-02	2.115989E-03	8.7101837E-01	1.1933764E
18	2.058919E-03	1.202759E-02	4.211089E-03	8.711437E-01	1.422844E
19	4.117813E-03	1.539011E-02	8.292313E-03	8.7138535E-01	5.229946E
20	8.235604E-03	3.561255E-02	1.922313E-02	8.718535E-01	1.562666E
21	1.647118E-02	4.954105E-02	2.008113E-02	8.7272235E-01	1.897188E
22	3.294235E-02	6.779840E-02	3.024674E-02	8.7423151E-01	2.992223E
23	6.588465E-02	8.975840E-02	1.3919479E-02	8.7631085E-01	1.882866E
24	1.317693E-01	1.112664E-01	2.8594779E-02	8.781085E-01	2.992223E
25	2.633533E-01	1.208997E-01	4.2386229E-02	8.752680E-01	1.2963005E
26	5.2507172E-01	1.009331E-01	5.6717794E-02	8.5782054E-01	2.963005E
27	1.054154E 00	4.974334E-02	5.6717794E-02	8.03208E-01	1.2963005E
28	2.108309E 00	8.543260E-02	3.780234E-02	7.037997E-01	1.6382207E
29	4.216618E 00	1.781907E-04	3.466037E-03	5.373893E-01	2.441114E
SUMS		7.211952E-01	2.785645E-01		2.1971152E

VALUES FOR HEAD/TAIL APPROXIMATION:

J1= 26 H1= 0.177246E-03 H2= 0.200000E 01 H3= 0.330000E-06 H4= 0.266667E 01





PROBABILITY VECTOR AND LINKS

	PROB	TABLE	NEXT	LAST
7	1.246153E-06	-1	0	0
8	1.417932E-06	-2	0	7
9	1.120355E-05	-3	0	8
10	1.366464E-05	-4	0	9
11	4.6515809E-05	-5	0	10
12	9.7797719E-04	-6	13	11
13	1.3578268E-04	-7	16	12
14	2.225602E-04	0	0	13
15	2.8496800E-04	-8	19	14
16	3.4742607E-04	-9	0	15
17	5.726512E-04	-10	22	16
18	8.091452E-04	-11	0	17
19	1.1809080E-03	-12	26	18
20	1.376381E-03	-13	0	19
21	1.5850522E-03	-14	27	20
22	1.801083E-03	-15	31	21
23	2.15761E-03	-16	0	22
24	3.464662E-03	-17	35	23
25	4.028697E-03	-18	0	24
26	5.1774E-03	-19	0	25
27	6.03999E-03	-20	39	26
28	7.832959E-03	-21	0	27
29	9.86647E-02	-22	0	28
30	1.3544839E-02	-23	40	29
31	1.5856369E-02	-24	0	30
32	2.225081E-02	-25	43	31
33	2.9417495E-02	-26	44	32
34	4.0617672E-02	-27	48	33
35	5.66633E-02	-28	0	34
36	7.832959E-02	-29	47	35
37	1.0986620E-01	-30	0	36
38	1.3544839E-01	-31	0	37
39	1.5856369E-01	-32	0	38
40	2.225081E-01	-33	0	39
41	2.9417495E-01	-34	0	40
42	4.0617672E-01	-35	0	41
43	5.66633E-01	-36	0	42
44	7.832959E-01	-37	0	43
45	1.0986620E-01	-38	0	44
46	1.3544839E-01	-39	0	45
47	1.5856369E-01	-40	0	46
	2.225081E-01	-41	0	47
	2.9417495E-01	-42	0	0
	4.0617672E-01	-43	0	0
	5.66633E-01	-44	0	0
	7.832959E-01	-45	0	0
	1.0986620E-01	-46	0	0
	1.3544839E-01	-47	0	0
	1.5856369E-01	-48	0	0
	2.225081E-01	-49	0	0
	2.9417495E-01	-50	0	0
	4.0617672E-01	-51	0	0
	5.66633E-01	-52	0	0
	7.832959E-01	-53	0	0
	1.0986620E-01	-54	0	0
	1.3544839E-01	-55	0	0
	1.5856369E-01	-56	0	0
	2.225081E-01	-57	0	0
	2.9417495E-01	-58	0	0
	4.0617672E-01	-59	0	0
	5.66633E-01	-60	0	0
	7.832959E-01	-61	0	0
	1.0986620E-01	-62	0	0
	1.3544839E-01	-63	0	0
	1.5856369E-01	-64	0	0
	2.225081E-01	-65	0	0
	2.9417495E-01	-66	0	0
	4.0617672E-01	-67	0	0
	5.66633E-01	-68	0	0
	7.832959E-01	-69	0	0
	1.0986620E-01	-70	0	0
	1.3544839E-01	-71	0	0
	1.5856369E-01	-72	0	0
	2.225081E-01	-73	0	0
	2.9417495E-01	-74	0	0
	4.0617672E-01	-75	0	0
	5.66633E-01	-76	0	0
	7.832959E-01	-77	0	0
	1.0986620E-01	-78	0	0
	1.3544839E-01	-79	0	0
	1.5856369E-01	-80	0	0
	2.225081E-01	-81	0	0
	2.9417495E-01	-82	0	0
	4.0617672E-01	-83	0	0
	5.66633E-01	-84	0	0
	7.832959E-01	-85	0	0
	1.0986620E-01	-86	0	0
	1.3544839E-01	-87	0	0
	1.5856369E-01	-88	0	0
	2.225081E-01	-89	0	0
	2.9417495E-01	-90	0	0
	4.0617672E-01	-91	0	0
	5.66633E-01	-92	0	0
	7.832959E-01	-93	0	0
	1.0986620E-01	-94	0	0
	1.3544839E-01	-95	0	0
	1.5856369E-01	-96	0	0
	2.225081E-01	-97	0	0
	2.9417495E-01	-98	0	0
	4.0617672E-01	-99	0	0
	5.66633E-01	-100	0	0







SIMULATION RESULTS FOR K= 0.10 AND SAMPLE SIZE = 30

THE FIRST FOUR MOMENTS ARE:

VARIANCE: 1.548733D C1 2.587971D C2 4.594324C C3 8.57C594D C4  
2.096324E-03 2.444954E CC 1.8C2241E C3 1.164885E C6

MEAN = 1.548733E 01  
VARIANCE = 1.894167E 01 STANDARD DEVIATION = 4.352202E 00  
SKEWNESS = -5.621020E-03  
KURTOSIS = 2.626644E 00

QUANTILES OF THE DISTRIBUTION:

ALPHA	ALPHA-QUANTILE	VARIANCE	NCRMALIZED QUANTILE
C.CC1	3.590595E	686967E--02	-2.751887
C.002	4.198548E	1.481914E--02	-2.611408
C.CC5	5.007994E	1.3774E--02	-2.424372
C.010	5.789165E	1.3750E--02	-2.243867
C.025	6.739015E	1.3692E--02	-2.024387
C.050	7.07254CE	1.36214E--02	-1.947320
C.100	8.292783E	1.35913E--02	-1.322177
C.200	1.167267E	1.355954E--02	-1.0884375
C.300	1.310211E	1.72559E--02	-0.5554076
C.400	1.43117CE	1.727170E--02	-0.2745778
C.500	1.547830E	1.727170E--02	-0.005077
C.600	1.666309E	1.727170E--02	0.2687522
C.700	1.789522E	1.727170E--02	0.555459
C.800	1.930099E	1.727170E--02	0.8782290
C.900	2.120237E	1.727170E--02	1.1655327
C.955	2.269212E	1.727170E--02	1.69223776
C.980	2.417113E	1.727170E--02	2.003624
C.990	2.509793E	1.727170E--02	2.217795
C.995	2.591594E	1.727170E--02	2.406757
C.999	2.6764C1E	1.727170E--02	2.602757
C.999	2.730328E	1.727170E--02	2.727366



SIMULATION RESULTS FOR K= 0.25 AND SAMPLE SIZE = 30

THE FIRST FOUR MOMENTS ARE:

VARIANCE:        1.549231D 01        2.490223D 02        4.137035D 03        7.080694D 04  
                  7.433707E-04        7.329798E-01        4.556223E 02        2.475938E 05

MEAN            = 1.549231E 01  
VARIANCE       = 9.011434E 00        STANDARD DEVIATION = 3.001905E 00  
SKEWNESS       = -4.122369E-03  
KURTCSIS       = 2.840780E 00

QUANTILES OF THE DISTRIBUTION:

ALPHA	ALPHA-QUANTILE	VARIANCE	NORMALIZED QUANTILE
0.001	6.525058E 00	5.840706E-02	-3.023095
0.002	7.202499E 00	2.115074E-02	-2.794908
0.005	7.921307E 00	1.086465E-02	-2.552787
0.010	8.647766E 00	4.937734E-03	-2.308087
0.020	9.385397E 00	3.008976E-03	-2.059627
0.025	9.645103E 00	3.421807E-03	-1.972147
0.050	1.053542E 01	1.831021E-03	-1.672254
0.100	1.158902E 01	4.436225E-03	-1.317364
0.200	1.292809E 01	1.696679E-03	-0.866317
0.300	1.389270E 01	1.344738E-03	-0.541399
0.400	1.472537E 01	2.145337E-03	-0.260755
0.500	1.549525E 01	1.542333E-03	-0.001599
0.600	1.626398E 01	9.771319E-04	0.257336
0.700	1.709277E 01	1.708522E-03	0.536495
0.800	1.806477E 01	1.431866E-03	0.863912
0.900	1.936656E 01	1.378786E-03	1.302402
0.950	2.043901E 01	3.398907E-03	1.663643
0.975	2.133118E 01	5.721221E-03	1.964158
0.980	2.160959E 01	6.842077E-03	2.057937
0.990	2.232549E 01	1.322764E-02	2.299078
0.995	2.305215E 01	2.466045E-02	2.543847
0.998	2.377327E 01	4.277430E-02	2.786746
0.999	2.423307E 01	5.257401E-02	2.940840





SIMULATION RESULTS FOR K= 0.50 AND SAMPLE SIZE = 30

THE FIRST FOUR MOMENTS ARE:

VARIANCE: 1.549263D 01 2.448627D 02 3.943646D 03 6.465488D 04  
 8.401840E-04 8.340271E-01 5.544180E 02 2.888450E 05

MEAN = 1.549263E 01  
 VARIANCE = 4.841497E 00 STANDARD DEVIATION = 2.200340E 00  
 SKEWNESS = 5.866662E-03  
 KURTOSIS = 2.899539E 00

QUANTILES OF THE DISTRIBUTION:

ALPHA	ALPHA-QUANTILE	VARIANCE	NORMALIZED QUANTILE
0.001	8.80285E 00	3.89128E-02	-3.059215
0.002	9.28968E 00	4.395074E-02	-2.070017
0.005	9.945611E 00	5.02623E-03	-2.056689
0.010	1.043447E 01	5.714556E-03	-2.040967
0.025	1.118998E 01	1.516437E-03	-2.085118
0.050	1.187312E 01	1.779421E-03	-1.676113
0.100	1.266379E 01	2.536774E-03	-1.310719
0.200	1.361239E 01	3.33225E-03	-0.872335
0.300	1.431863E 01	4.445571E-03	-0.545555
0.400	1.492431E 01	5.67719E-04	-0.266046
0.500	1.548894E 01	6.67361E-04	-0.020511
0.600	1.606097E 01	7.78617E-04	0.259247
0.700	1.666231E 01	9.074037E-04	0.537456
0.800	1.736612E 01	1.078440E-03	0.861028
0.900	1.832162E 01	1.331547E-03	1.303744
0.950	1.911794E 01	1.688307E-02	1.697191
0.975	1.999076E 01	2.046503E-02	1.984694
0.990	2.056076E 01	2.50966E-02	2.227679
0.995	2.109485E 01	3.008556E-02	2.338764
0.999	2.173523E 01	4.184528E-02	2.558162
0.999	2.217344E 01	5.99999E-02	3.061504



SIMULATION RESULTS FOR K= 0.75 AND SAMPLE SIZE = 30

THE FIRST FOUR MOMENTS ARE:

VARIANCE: 1.550756D 01 2.437767D 02 3.882414D 03 6.261050D 04  
 3.363851E-04 3.336535E-01 1.948461E 02 8.956469E 04

MEAN = 1.550756E 01  
 VARIANCE = 3.292473E 00 STANDARD DEVIATION = 1.814517E 00  
 SKEWNESS = -1.425088E-02  
 KURTOSIS = 2.959830E 00

QUANTILES OF THE DISTRIBUTION:

ALPHA	ALPHA-QUANTILE	VARIANCE	NORMALIZED QUANTILE
0.001	9.888675E 00	1.059276E-02	-3.142752
0.002	1.025604E 01	2.259400E-02	-2.937002
0.005	1.085379E 01	1.066357E-02	-2.602219
0.010	1.131085E 01	9.343445E-03	-2.346232
0.020	1.176631E 01	4.140753E-03	-2.091141
0.025	1.194138E 01	1.512869E-03	-1.993084
0.050	1.252072E 01	1.957431E-03	-1.668615
0.100	1.316931E 01	1.416034E-04	-1.305357
0.200	1.397234E 01	5.245046E-04	-0.855603
0.300	1.455316E 01	6.248999E-04	-0.527498
0.400	1.505430E 01	8.297206E-04	-0.249624
0.500	1.551409E 01	7.378636E-04	0.007893
0.600	1.597451E 01	5.092884E-04	0.265759
0.700	1.646059E 01	9.245067E-04	0.537999
0.800	1.704263E 01	4.539427E-04	0.863987
0.900	1.783040E 01	5.107326E-04	1.305193
0.950	1.849435E 01	5.730530E-04	1.677056
0.975	1.905101E 01	2.615529E-03	1.988825
0.980	1.919690E 01	3.075583E-03	2.070533
0.990	1.969147E 01	3.800856E-03	2.347528
0.995	2.016524E 01	4.898641E-03	2.612873
0.998	2.066853E 01	3.832733E-03	2.894755
0.999	2.104907E 01	1.654354E-02	3.107885



SUBROUTINE GMINIT(K,BETA)

ADDITIONAL ENTRY POINT:  
GAMA(Z,IX)

PURPOSE:

GENERATION OF GAMMA RANDOM DEVIATES WITH SHAPE  
PARAMETER LESS THAN ONE.

PROGRAMMER:

D.W. ROBINSON, LT, USN

METHOD:

A MODIFICATION OF MARSAGLIA'S RECTANGLE-WEDGE-TAIL  
METHOD FOR NORMAL DEVIATES IS USED. THE GAMMA PDF  
IS DECOMPOSED INTO A HEAD REGION, A NUMBER  
(DEPENDENT ON K) OF RECTANGLES AND WEDGES AND A TAIL  
REGION. THE GMINIT SECTION OF THE SUBROUTINE ALSO  
SETS UP A BINARY SEARCH TREE TO BE USED FOR  
EFFICIENT SELECTION OF THE PROPER REGION DURING THE  
ACTUAL GENERATING PROCESS, WHICH IS HANDLED BY THE  
GAMA SECTION.

DESCRIPTION OF PARAMETERS:

K GAMMA DISTRIBUTION SHAPE PARAMETER (MUST BE  
.GE. 0.05 AND .LT. 1.0) (REAL\*4)  
BETA GAMMA DISTRIBUTION SCALE PARAMETER (REAL\*4)  
IX SEED FOR UNIFORM RANDOM NUMBER GENERATOR  
Z RETURNED GAMMA DEVIATE (REAL\*4)

NOTE THAT THE DENSITY FUNCTION OF A GAMMA RANDOM  
VARIABLE IS GIVEN BY:

$$F(X) = (1/BETA)^K X^{K-1} E^{(-X/BETA)} / GAMMA(K)$$

THE FOLLOWING SUBROUTINES ARE CALLED:

IGAM(K,X) COMPUTES THE INCOMPLETE GAMMA  
FUNCTION (GAMMA CDF).  
INVGM(K,X) COMPUTES THE INVERSE GAMMA CDF  
RANDOM(IX,U,N) RETURNS A VECTOR U OF N UNIFORM  
RANDOM NUMBERS.  
OVFLOW SET-UP ENTRY POINT FOR RANDOM

NOTE:

UNDERFLOW IS POSSIBLE WHEN K IS LESS THAN .18 AND  
BECOMES MORE LIKELY AS K DECREASES. WHEN K IS .05  
THE PROBABILITY OF UNDERFLOW IS ABOUT .00013 FOR  
ANY GENERATED DEVIATE.

SUBROUTINE GMINIT (K,BETA)

REAL\*4 K, IGAM, INVGM  
INTEGER\*4 FIRST, TABLE, BOTTOM, END  
LOGICAL\*1 USED

DIMENSION X(101), H(100), P(100), Q(100), R(100), B(100)  
DIMENSION PROB(202), TABLE(202), NEXT(202), LAST(202)  
DIMENSION TEST(202), LIST(202), USED(202)  
DIMENSION RAND(2)  
EQUIVALENCE (U,RAND(1)), (V,RAND(2))



THIS FIRST SECTION INITIALIZES CONSTANTS AND TABLES TO BE USED BY GAMA WHEN IT IS CALLED. THE FOLLOWING ARE TO BE CALCULATED:

FIRST STARTING POINT FOR BINARY SEARCH  
 NEXT, LINK VECTORS FOR BINARY SEARCH PROCEDURE  
 LAST  
 PROB VECTOR OF CUMULATIVE PROBABILITIES  
 TABLE USED TO LOOK-UP RESULT OF SEARCH IN PROB  
 J1 POSITION IN PROB OF P0  
 X(I) LEFT-HAND BOUNDARY OF I-TH RECTANGLE  
 H(I) WIDTH OF I-TH RECTANGLE  
 R(I) REJECTION RATIO FOR I-TH WEDGE  
 B(I) Y ORDINATE FOR I-TH WEDGE  
 ALPHA K-1  
 P0 PROBABILITY FOR HEAD REGION  
 PN PROBABILITY FOR TAIL REGION  
 H1 TO CONSTANTS FOR HEAD APPROXIMATION TO INVERSE  
 H4 GAMMA CDF

CHECK FOR K IN RANGE  
 IF((K .GE. 0.05) .AND. (K .LT. 1.0)) GO TO 5

WRITE(6,4)K  
 4 FORMAT(/'OGMINIT CALLED WITH K=',1PE16.6,  
 \* ' OUT OF RANGE'/)  
 RETURN

GET UPPER BOUND ON NUMBER OF RECTANGLES REQUIRED

5 N = 20. + 6.6/K  
 IF(N .GT. 100) N = 100

INITIALIZE CONSTANTS

M = 2\*N + 2  
 MM = M - 1  
 ALPHA = K - 1.  
 GK = GAMMA(K)  
 P0 = 5.0E-5 / (K \* K)  
 HFAC = 2.

FIND CONSTANTS FOR HEAD APPROXIMATION

H1 = K \* GK  
 H2 = 1. / K  
 H3 = 2.2E-7 \* (K + 1.)  
 H4 = 4. / (K + 1.)

SET UP RECTANGLE BOUNDS

X(1) = (H1 \* P0) \*\* H2  
 X(1) = 2. \* X(1) / (1. + SQRT(1. - H4 \* X(1)))  
 P0 = IGAM(K,X(1))  
 H1 = H1 \* P0  
 H(1) = .25 \* X(1)

DO 10 I=2,N  
 X(I) = X(I-1) + H(I-1)  
 H(I) = H(I-1) \* HFAC  
 P(I) = 0.  
 Q(I) = 0.

10 CONTINUE  
 X(N+1) = X(N) + H(N)

ZERO PROBABILITY VECTORS AND LINKS

DO 15 I=1,M  
 NEXT(I) = 0  
 LAST(I) = 0  
 PROB(I) = 0  
 LIST(I) = 0





```

      USED(I) = .FALSE.
15  CONTINUE

C
C
C  FIND COEFFICIENTS FOR NEWTON-RAPHSON APPROXIMATION
C  TO DENSITY FUNCTION TANGENT
C
      B1 = -2. * ALPHA
      B2 = ALPHA * (ALPHA - 1.)
      A1 = B1 + 1.
      A2 = ALPHA * (ALPHA - 2.)
      C = 1. - ALPHA

C
C  FIND RECTANGLE PROBABILITIES AND WEDGE VALUES
C
      PL = P0
      FL = EXP( ALPHA * ALOG(X(1)) - X(1)) / GK
      DO 40 I=1,N
          FU = EXP( ALPHA * ALOG(X(I+1)) - X(I+1)) / GK
          PU = IGAM(K,X(I+1))
          P(I) = H(I) * FU
          Q(I) = PU - PL - P(I)

C
C
C  NEWTON-RAPHSON ITERATION TO FIND POINT WHERE
C  TANGENT TO PDF IS PARALLEL TO CHORD
C
      W = X(I)
      S = (FU - FL) / H(I)
      SC = S * GK
      DO 20 J=1,15
          *      Y = W * ((W + A1) * W + A2 +
          *          SC * EXP(C * ALOG(W) + W))
          *          / ((W + B1) * W + B2)
          IF(ABS(Y - W) .LT. 1.E-4 * Y) GO TO 30
          W = Y
20  CONTINUE

C
C
C  FIND VALUES FOR REJECTION METHOD
C
30  B(I) = EXP(ALPHA * ALOG(Y) - Y)/GK + S*(X(I) - Y)
      R(I) = (B(I) - FU) / (FL - FU)

C
C
C  TEST TO SEE IF ENOUGH RECTANGLES HAVE BEEN TAKEN
C
      IF(PU .GT. 0.999) GO TO 45

C
C
C  RESET PROBABILITY AND FUNCTION VALUES FOR
C  NEXT RECTANGLE
C
      PL = PU
      FL = FU
40  CONTINUE

C
C
C  FIND LOWER END INDEX FOR PROB
C
45  LOW = 2 * (N-I) + 1

C
C
C  FIND TAIL PROBABILITY
C
      PN = 1. - PU

C
C
C  GENERATE PROBABILITY VECTOR
C
      DO 80 I=1,N
          PROB(I) = P(I)
          PROB(I+N) = Q(I)
          TABLE(I) = I
          TABLE(I+N) = -I
80  CONTINUE
      PROB(M-1) = P0
      PROB(M) = PN
      TABLE(M-1) = 0
      TABLE(M) = 0

```



```

C
C
C      SORT PROBABILITY VECTOR
C
C      DO 110 I=1,MM
C          ICH = 0
C          L = M - I
C          DO 100 J=1,L
C              IF(PROB(J) - PROB(J+1)) 100,100,90
C              TEMP = PROB(J)
C              PROB(J) = PROB(J+1)
C              PROB(J+1) = TEMP
C              ITEMP = TABLE(J)
C              TABLE(J) = TABLE(J+1)
C              TABLE(J+1) = ITEMP
C              ICH = 1
C          CONTINUE
C          IF(ICH) 120,120,110
C      110 CONTINUE
C
C      CCONVERT PROB TO CUMULATIVE PROBABILITIES
C      FIND FIRST AND J1
C
C      120 J1 = 0
C          FIRST = 0
C          L = LOW + 1
C          DO 130 I=L,M
C              IF((TABLE(I) .EQ. 0) .AND. (PROB(I) .EQ. P0))J1=I
C              PROB(I) = PROB(I) + PROB(I-1)
C              IF((PROB(I) .GE. .5) .AND. (FIRST .EQ. 0))FIRST=I
C      130 CONTINUE
C          IF(FIRST .EQ. M) FIRST = MM
C          PROB(M) = 1.
C          IF(J1 .EQ. 0) J1 = LOW
C
C      NOW DETERMINE THE VECTORS NEXT AND LAST
C
C      150 BOTTOM = 1
C          END = 1
C          PR = .25
C          LIST(1) = FIRST
C          TEST(1) = .5
C          USED(FIRST) = .TRUE.
C
C      FIND NEXT(K) AND LAST(K) FOR EACH K ON LIST
C
C      151 DO 159 I=1,BOTTOM
C          LI = LIST(I)
C
C          FIND NEXT(LI)
C
C          IF LI+1 HAS BEEN TESTED GO ON TO FIND LAST(LI)
C          IF(USED(LI+1)) GO TO 155
C
C          GET PROBABILITY VALUE FOR NEXT(LI)
C          PRN = TEST(I) + PR
C
C          FIND J SUCH THAT PROB(J) > PRN
C          DO 152 J=LOW,MM
C              IF(PROB(J) .GT. PRN) GO TO 153
C      152 CONTINUE
C
C          GET K SUCH THAT PROB(K) HAS NOT BEEN TESTED AND
C          LI .LT. K .LE. J. IF K EXISTS, SET NEXT(LI)=K.
C
C      153 IF( .NOT.USED(J)) GO TO 154
C          J = J - 1
C          IF(LI - J) 153,155,155
C      154 NEXT(LI) = J
C
C

```



```

C      NEXT(LI) HAS NOW BEEN FOUND IF IT IS NCN-ZERO.
C      NOW ADD IT TO THE LIST IN PLACE OF LI FOR THE
C      NEXT PASS THROUGH THE LIST.
155    LIST(I) = NEXT(LI)
C      NOW FIND LAST(LI)
C      IF LI = LOW CR PROB(LI-1) HAS BEEN USED, GO
C      ON TO THE NEXT VALUE ON THE LIST
C      IF((LI .EQ. LOW) .OR. USED(LI-1)) GO TO 159
C      GET PROBABILITY VALUE FOR LAST(LI)
C      PRL = TEST(I) - PR
C      RESET PROBABILITY TEST VALUE FOR NEXT PASS
C      TEST(I) = PRN
C      FIND J SUCH THAT PROB(J) > PRL
C      DO 156 J=LOW,MM
C          IF(PROB(J) .GT. PRL) GO TO 157
156    CONTINUE
C      FIND K SUCH THAT PROB(K) HAS NOT BEEN TESTED AND
C      LI .GT. K .GE. J. IF K EXISTS, SET LAST(LI)=K.
157    IF( .NOT.USED(J)) GO TO 1571
C      J = J + 1
C      IF(LI - J) 158,158,157
1571   LAST(LI) = J
C      GO TO 1585
158    J = LI
C      LAST(LI) HAS NOW BEEN FOUND. ADD IT TO THE END OF
C      THE LIST FOR THE NEXT PASS.
1585   END = END + 1
C      LIST(END) = J
C      TEST(END) = PRL
159    CONTINUE
C      NOW RESET THE LIST FOR THE NEXT PASS BY ELIMINATING
C      ZERO ENTRIES.
C      BOTTOM = END
C      PR = PR * .5
C      I = 1
C      TEST FOR ZERO ENTRY
1591   IF(LIST(I)) 160,160,163
C      ZERO ENTRY FOUND. ADJUST BOTTOM OF LIST
160    BOTTOM = BOTTOM - 1
C      IF LIST EMPTY, QUIT.
C      IF(BOTTOM) 165,165,161
C      SHIFT OTHER LIST ENTRIES UP
161    DO 162 J=I,BOTTOM
C          LIST(J) = LIST(J+1)
C          TEST(J) = TEST(J+1)
162    CONTINUE
C      GO TO 1591
C

```



```

C      SET USED FLAG FOR LIST ENTRY
C
163 USED(LIST(I)) = .TRUE.
C
C      GET NEXT LIST ENTRY
C
      I = I + 1
      IF (I .LE. BOTTOM) GO TO 1591
C
C      DONE WITH LIST RESETTING.  GO BACK FOR ANOTHER PASS
C
      END = BOTTOM
      GC TO 151
C
C      SETUP FIRST CALL TO RANDOM
C
165 CALL OVFLOW
C
C      THE FOLLOWING STATEMENTS PRINT OUT A TABULAR LISTING
C      OF THE RESULTS OF GMINIT.  THEY MAY BE REMOVED, IF
C      DESIRED, IN A PRODUCTION RUN.
C
      WRITE(6,170)K,BETA
170 FORMAT('1GENERATED VALUES FOR K=',1PE14.6,
*        BETA=',E14.6//)
      WRITE(6,175)PO,PN
175 FORMAT('OHEAD PROBABILITY=',1PE14.6,
*        'TAIL PROBABILITY=',E14.6//)
      WRITE(6,180)
180 FORMAT('ORECTANGLE/WEDGE VALUES'//2X,'I',9X,'X(I)',
*        12X,'P(I)',12X,'Q(I)',12X,'R(I)',12X,'B(I)'//)
C
C      ADD UP TOTAL RECTANGLE AND WEDGE PROBABILITIES AND
C      PRINT OUT RESULTS
C
      SUM1 = 0.
      SUM2 = 0.
      DC 192 I=1,N
C
C      NO ACTION FOR ZERO PROBABILITIES
C
      IF(P(I)) 193,193,185
C
185      SUM1 = SUM1 + P(I)
      SUM2 = SUM2 + Q(I)
      WRITE(6,190) I,X(I),P(I),Q(I),R(I),B(I)
190      FORMAT(1X13,1P5E16.6)
192 CONTINUE
C
193 WRITE(6,194)SUM1,SUM2
194 FORMAT('OSUMS',15X,1P2E16.6)
C
C      WRITE OUT CONSTANTS
C
      WRITE(6,195)J1,H1,H2,H3,H4
195 FORMAT('/OVALUES FOR HEAD/TAIL APPROXIMATION:'//
*        'J1=',14/'H1=',E16.6,'H2=',E16.6,
*        'H3=',E16.6,'H4=',E16.6)
C
C      WRITE OUT BINARY SEARCH DATA
C
      WRITE(6,196)FIRST
196 FORMAT('/OSTARTING POINT FOR BINARY SEARCH',14)
      WRITE(6,197)(I,PROB(I),TABLE(I),NEXT(I),LAST(I),
*        I=LOW,M)
197 FORMAT('/OPROBABILITY VECTOR AND LINKS'//
*        14X,'PROB',9X,'TABLE',4X,'NEXT',5X,
*        'LAST'//((1X115,1PE16.6,OP319))
C
C      FINISHED WITH SET-UP PHASE.  QUIT.
C
      RETURN

```





```

C
C      THIS IS THE SECTION WHICH ACTUALLY GENERATES THE
C      GAMMA VARIATES.
C      ENTRY  GAMA(IX,Z)
C      GET TWO UNIFORM RANDOM VARIATES
C      CALL RANDOM(IX,U,2)
C      CONDUCT BINARY SEARCH USING THE FIRST UNIFORM VARIATE
C      J = FIRST
200  IF(U - PROB(J)) 210,250,230
C      U < CURRENT VALUE.  USE LAST FOR FOLLOWING TEST.
C      210 IF(LAST(J)) 250,250,220
220  J = LAST(J)
      GO TO 200
C      U > CURRENT VALUE.  USE NEXT FOR FOLLOWING TEST.
230  IF(NEXT(J)) 245,245,240
240  J= NEXT(J)
      GO TO 200
C      245 J= J + 1
C      LOCATED PROPER PROBABILITY DIVISION.  LOOK UP IN TABLE
250  N = TABLE(J)
      IF(N) 260,290,320
C      TABLE VALUE < 0.  SAMPLE FROM N-TH WEDGE.
260  N= -N
C      GET ONE MORE UNIFORM DEVIATE
C      CALL RANDOM(IX,U,1)
C      GET U .LE. V
270  IF(U .LE. V) GO TO 280
      TEMP = U
      U = V
      V = TEMP
C      GET TRIAL GAMMA VALUE
280  Z = X(N) + H(N)*U
C      PERFORM REJECTION TEST
      IF(V .LE. R(N)) GO TO 330
C      FIRST TEST FAILED.  GET VALUE FOR SECOND TEST.
      W = U + EXP(ALPHA * ALOG(Z) - Z) / B(N)
      IF(V .LE. W) GO TO 330
C      SECOND TEST ALSO FAILED.  REPEAT LOOP WITH TWO
C      NEW UNIFORM VARIATES
      CALL RANDOM(IX,U,2)
      GC TO 270
C
C      TABLE VALUE = 0.  USE HEAD/TAIL DISTRIBUTIONS.

```



```

C      DETERMINE WHICH DISTRIBUTION TO USE.
C
C 290 IF(J .EQ. J1) GO TO 300
C
C      USE TAIL DISTRIBUTION.
C
C      Z = INVGAM(K, PN * V)
C      GO TO 330
C
C      USE HEAD DISTRIBUTION.
C
C 300 Z = (H1 * V) ** H2
C      IF(Z .LT. H3) GO TO 330
C      Z = 2. * Z / (1. + SQRT(1. - H4 * Z))
C      GO TO 330
C
C      TABLE VALUE > 0.  SAMPLE FROM N-TH RECTANGLE.
C
C 320 Z = X(N) + H(N) * V
C
C      Z IS NOW A UNIT GAMMA VARIATE.  SCALE IT AND EXIT.
C
C 330 Z = Z * BETA
C      RETURN
C      END

```



GAMA	CSECT		
	ENTRY	START	
BASE1	EQU	2	
BASE2	EQU	3	
SAVE	EQU	4	
REGA	EQU	5	
REGB	EQU	6	
REGC	EQU	8	
SRCH	EQU	7	
*			
	STM	14,REGC,12(13)	SAVE CALLING PROGRAM REGS
	BALR	BASE1,0	SET UP BASE REGISTERS
	USING	*,BASE1	
	LA	BASE2,DATA	
	USING	DATA,BASE2	
	LR	SAVE,13	SAVE ADDRESS OF CALLING
*			PROGRAM SAVE AREA
	LA	13,SVAREA	LOAD OWN SAVE AREA ADDRESS
	ST	13,8(0,SAVE)	STORE OWN SAVE AREA ADDRESS
	ST	SAVE,4(0,13)	STORE CALLING SAVE AREA
*			ADDRESS
	LA	1,ARG1	LINK TO RANDOM FOR TWO
	L	15,=V(RANDOM)	UNIFORM DEVIATES
	BALR	14,15	
*			
	L	REGB,FIRST	START BINARY SEARCH
	L	SRCH,U	LOAD UNIFORM FOR SEARCH
	BALR	REGC,0	LOAD BRANCH ADDRESS
*			
SEARCH	LR	REGA,REGB	LOAD NEXT INDEX
	C	SRCH,PROB(REGA)	COMPARE
	BC	4,LOW	
	BC	8,FOUND	
*			
HIGH	L	REGB,NEXT(REGA)	GET NEXT INDEX FROM "NEXT"
	BCTR	REGB,REGC	BRANCH BACK IF INDEX
*			IS NON-ZERO
	LA	REGA,4(REGA)	
	BC	15,FOUND	THROUGH WITH SEARCH
*			
LOW	L	REGB,LAST(REGA)	GET NEXT INDEX FROM "LAST"
	BCTR	REGB,REGC	BRANCH BACK IF NON-ZERO
*			
FOUND	L	REGB,TABLE(REGA)	FIND WHICH METHOD TO USE
	LTR	REGB,REGB	
	BC	2,RECT	RECTANGLE IF TABLE
*			VALUE GREATER THAN 0
*			
	STD	2,SAVED	SAVE FLOATING POINT REG 2
	BC	8,TAIL	USE HEAD/TAIL IF TABLE
*			VALUE = 0
*			
WEDGE	LCR	REGB,REGB	USE WEDGE METHOD IF TABLE
*			VALUE < 0
*			
	LA	1,ARG2	GET NEW RANDOM NUMBER
	L	15,=V(RANDOM)	
	BALR	14,15	
*			
W1	LE	0,U	GET U LESS THAN OR EQUAL V
	CE	0,V	
	BC	12,W2	
*			
	LE	2,V	EXCHANGE U AND V
	STE	0,V	
	LER	0,2	
	STE	0,U	
*			
W2	ME	0,H(REGB)	FIND Z = X(N) + U * H(N)
	AE	0,X(REGB)	
*			



	LE	2,V	TEST IF V IS LESS THAN OR
	CE	2,R(REGB)	EQUAL TO R(N)
	BC	12,DONE	PASS REJECTION TEST IF SO
*	STE	0,Z	THIS STEP IS REACHED ONLY
			1/3 OF THE TIME
	LA	1,ARG3	FIND ALOG(Z)
	L	15,=V(ALOG)	
*	BALR	14,15	
	ME	0,ALPHA	
	SE	0,Z	
*	STE	0,ZP	$ZP = (K - 1) * ALOG(Z) - Z$
	LA	1,ARG3A	FIND $EXP(ZP) =$
	L	15,=V(EXP)	$EXP(-Z) * (Z ** K-1)$
*	BALR	14,15	
	DE	0,B(REGB)	FIND $W = U +$
	AE	0,U	$EXP(-Z) * (Z ** K-1) / B(N)$
	CE	0,V	TEST IF V IS LESS THAN OR
	LE	0,Z	EQUAL TO W
*	BC	2,DONE	QUIT IF SO
	LA	1,ARG1	OTHERWISE GET TWO NEW
	L	15,=V(RANDOM)	UNIFORM DEVIATES
	BALR	14,15	
	BC	15,W1	AND REPEAT THE LOOP
*			
TAIL	C	REGA,J1	SEE IF HEAD OR TAIL IS
	BC	8,HEAD	REQUIRED
	LE	0,PN	TAIL WILL BE USED.
	ME	0,V	$Z = INVGAM(K, PN*V)$
	STE	0,Z	
	LA	1,ARG4	
	L	15,=V(INVGAM)	
	BALR	14,15	
	BC	15,DONE	
*			
HEAD	LE	0,V	COMPUTE FIRST ORDER APPROX
	ME	0,H1	$(V * PO * GAMMA(K+1) )$
	STE	0,Z	$** 1/K$
	LA	1,ARG5	
	L	15,=V(FRXPR#)	
	BALR	14,15	
	CE	0,H3	SEE IF SECOND ORDER APPROX
	BC	4,DONE	IS NECESSARY
	STE	0,Z	
	ME	0,H4	$Z = 2*Z / (1 +$
	LE	2,ONEFL	$SQRT(1 - 4*Z/(K+1))) )$
	SER	2,0	
	STE	2,H5	
	LA	1,ARG6	
	L	15,=V(SQRT)	
	BALR	14,15	
	AE	0,ONEFL	
	LE	2,Z	
	ME	2,TWOFL	
	DER	2,0	
	LER	0,2	
*			
DONE	LD	2,SAVED	RESTORE FLOATING POINT
*			REGISTER 2
	BC	15,SCALE	
*			
RECT	LE	0,H(REGB)	$Z = X(N) + V * H(N)$
	ME	0,V	
	AE	0,X(REGB)	
*			
SCALE	ME	0,BETA	SCALE VARIATE AND EXIT
*			
	LR	13,SAVE	RESTORE OLD SAVE AREA
*			ADDRESS





```

*      LM      2,REGC,28(13)      RESTORE OLD GENERAL
*      L        14,12(13)          PURPOSE REGISTERS
*      MVI      12(13),X'FF'      RESTORE RETURN ADDRESS
*      BCR      15,14              SET END OF SUBROUTINE FLAG

*
*      DROP    BASE1
*      DROP    BASE2

*
*      ENTRY POINT FOR INITIALIZING ROUTINE

START  STM      14,REGC,12(13)      SAVE GP REGISTERS
BALR   BASE1,0                      SET UP BASE REGISTERS
USING  *,BASE1
LA     BASE2,DATA
USING  DATA,BASE2
LR     SAVE,13
LA     13,SVAREA
ST     13,8(0,SAVE)
ST     SAVE,4(0,13)
L      REGA,0(1)
MVC    K(4),0(REGA)                GET SET UP ARGUMENTS
L      REGA,4(1)                   K (SHAPE PARAMETER)
MVC    BETA(4),0(REGA)             BETA (SCALE PARAMETER)
L      REGA,8(1)
MVC    IX(4),0(REGA)              IX (RANDOM NUMBER SEED)

*
LA     1,ARG7
L      15,=V(GMINIT)              CALL GMINIT
BALR   14,15

*
L      15,=V(OVFLOW)              CALL OVFLOW TO SET UP FOR
BALR   14,15                      RANDCM

*
LR     13,SAVE
LM     2,REGC,28(13)
L      14,12(13)
MVI    12(13),X'FF'
BCR    15,14

*
*
IX      DS      0D
U       DC      4X'00'
V       DC      4X'00'
Z       DC      4X'00'
ZP      DC      4X'00'
ONE     DC      F'1'
TWO     DC      F'2'
ONEFL   DC      E'1.0'
TWOFL   DC      E'2.0'
*
K       DC      4X'00'
BETA    DC      4X'00'
*
ALPHA   DC      4X'00'
FIRST   DC      4X'00'
PROB    DC      4X'00'
P1      DC      808X'00'
NEXT    DC      4X'00'
N1      DC      808X'00'
LAST    DC      4X'00'
L1      DC      808X'00'
TABLE   DC      4X'00'
T1      DC      808X'00'
*
DATA    DS      0F
X        DC      4X'00'
X1       DC      404X'00'
H        DC      4X'00'
HP       DC      400X'00'
R        DC      4X'00'

```



R1	DC	400X'00'
B	DC	4X'00'
B1	DC	400X'00'
J1	DC	4X'00'
PN	DC	4X'00'
H1	DC	4X'00'
H2	DC	4X'00'
H3	DC	4X'00'
H4	DC	4X'00'
H5	DC	4X'00'

*		
ARG1	DC	A(IX)
	DC	A(U)
	DC	X'80'
	DC	AL3(TWO)
ARG2	DC	A(IX)
	DC	A(U)
	DC	X'80'
	DC	AL3(ONE)
ARG3	DC	X'80'
	DC	AL3(Z)
ARG3A	DC	X'80'
	DC	AL3(ZP)
ARG4	DC	A(K)
	DC	X'80'
	DC	AL3(Z)
ARG5	DC	A(Z)
	DC	X'80'
	DC	AL3(H2)
ARG6	DC	X'80'
	DC	AL3(H5)
ARG7	DC	A(K)
	DC	A(ALPHA)
	DC	A(FIRST)
	DC	A(P1)
	DC	A(N1)
	DC	A(L1)
	DC	A(T1)
	DC	A(X1)
	DC	A(HP)
	DC	A(R1)
	DC	A(B1)
	DC	A(J1)
	DC	A(PN)
	DC	A(H1)
	DC	A(H2)
	DC	A(H3)
	DC	X'80'
	DC	AL3(H4)

*		
SVAREA	DS	18F
SAVED	DS	3D
	END	



```

FUNCTION IGAM(K,X)
C
C
C PURPOSE:
C
C EVALUATION OF THE INCOMPLETE GAMMA FUNCTION:
C
C      K-1 -Y
C      INTEGRAL OF Y  E  FROM 0 TO X DIVIDED BY
C      GAMMA(K)
C
C METHOD:
C
C FOR VALUES OF X LESS THAN 5 AN INFINITE SERIES
C APPROXIMATION IS USED; FOR X GREATER THAN 5 A
C CONTINUED FRACTION APPROXIMATION IS USED. IF X
C IS LESS THAN OR EQUAL TO ZERO, ZERO IS RETURNED.
C
C
C      IMPLICIT REAL*8(D)
C      REAL*4 IGAM,K,V(6)
C      REAL*8 EPS/1.D-13/
C
C      TEST VALUE OF X
C      IF(X .GT. 0.)GO TO 10
C
C      X .LE. 0. RETURN IGAM = 0.
C      IGAM=0.
C      RETURN
C
C      DECIDE WHICH APPROXIMATION TO USE
10  IF(X .GT. 5.0)GO TO 50
C
C      USE INFINITE SERIES APPROXIMATION
C      DX=DBLE(X)
C      DK=DBLE(K)
C
C      INITIALIZE SUM AND TERM
C      DTERM=DX**DK/DGAMMA(DK)
C      DSUM=DTERM/DK
C
C      SUM THE SERIES
C      DO 30 I=1,30
C
C      TEST FOR CONVERGENCE
C      IF(DABS(DTERM)-EPS*DSUM)40,40,20
C
C      HASN'T CONVERGED. ADD ANOTHER TERM
20  DTERM=DTERM*(-DX)/DFLOAT(I)
C      DSUM=DSUM+DTERM/(DK+DFLOAT(I))
30  CONTINUE
C
C      SERIES CONVERGED. EXIT.
40  IGAM=SNGL(DSUM)
C      RETURN
C
C
C      USE CONTINUED FRACTION APPROXIMATION
C
C      INITIALIZE CONTINUED FRACTION VALUES
50  Y=1.-K
C      W=X+Y+1.0
C
C      SET UP RATIO VECTOR
C      V(1)=1.0
C      V(2)=X
C      V(3)=X+1.0
C      V(4)=W*X
C
C      R=V(3)/V(4)
C      CCUNT=1.0
C
C      GET NEXT APPROXIMATION
60  Y=Y+1.0

```



```

W=W+2.0
V(5)=V(3)*W-V(1)*Y*COUNT
V(6)=V(4)*W-V(2)*Y*COUNT
RATIO=V(5)/V(6)

```

```

C
C TEST FOR CONVERGENCE
C IF(ABS(RATIO-R) .LE. 1.0E-6*R)GO TO 90
C
C DIDN'T CONVERGENCE. SET UP FOR NEXT ITERATION.

```

```

R=RATIO
CCUNT=CCUNT+1.0
DO 70 I=1,4
V(I)=V(I+2)
70 CONTINUE

```

```

C
C TEST IF SCALING REQUIRED
C IF(V(5) .LT. 1.0E 35)GO TO 60
C

```

```

C SCALE DOWN RATIO VECTOR
DO 80 I=1,4
V(I)=V(I)*1.0E-35
80 CONTINUE
GO TO 60

```

```

C
C CCNVERGED. CONVERT RESULT AND EXIT.
90 IGAM = 1.-(RATIO*EXP(K*ALOG(X)-X)/GAMMA(K))
RETURN
END

```





```

FUNCTION  INVGAM(K,Z)
C
C      THIS FUNCTION SUBPROGRAM SOLVES THE EQUATION
C       $G(K;X) = 1 - Z$ 
C      BY NEWTON-RAPHSON ITERATION, WHERE G(K;X) IS THE
C      INCOMPLETE GAMMA FUNCTION WITH SHAPE PARAMETER K LESS
C      THAN 1.0.
C
C      REAL*4 K,INVGAM,KP,KBAR
C      DATA EPS/1.0E-6/
C      DIMENSION V(6)
C
C      TEST FOR Z IN RANGE
C      IF(Z .GT. 0.) GO TO 10
C      INVGAM=0.
C      RETURN
C
C      TEST IF NEW K VALUE
10  IF(K .EQ. KP) GO TO 20
C
C      NEW VALUE.  RECOMPUTE K-DEPENDENT CONSTANTS.
C      KP=K
C      KBAR=1.-K
C      GK=GAMMA(K)
C
C      INITIALIZE ITERATION VALUES
20  ZBAR=-Z*GK
C      X=1.0
C
C      MAIN ITERATION LOOP.
C
C      INITIALIZE CONTINUED FRACTION VALUES.
30  Y=KBAR
C      W=X+Y+1.0
C      V(1)=1.0
C      V(2)=X
C      V(3)=X+1.0
C      V(4)=W*X
C      R=V(3)/V(4)
C      CCUNT=1.0
C
C      CONTINUED FRACTION APPROXIMATION FOR G(K;X)
40  Y=Y+1.0
C      W=W+2.0
C      V(5)=V(3)*W-V(1)*Y*COUNT
C      V(6)=V(4)*W-V(2)*Y*COUNT
C      RATIO=V(5)/V(6)
C
C      TEST FOR CONVERGENCE
C      IF(ABS(RATIO-R) .LE. EPS*R) GO TO 70
C
C      DIDN'T CONVERGE.  SET UP FOR NEXT ITERATION.
C      R=RATIO
C      CCUNT=COUNT+1.0
C      DO 50 I=1,4
C      V(I)=V(I+2)
50  CONTINUE
C
C      TEST IF SCALING REQUIRED
C      IF(V(5) .LT. 1.0E 35)GO TO 40
C
C      SCALE DOWN V VECTOR TO PREVENT OVERFLOW.
C      DC 60 I=1,4
C      V(I)=V(I)*1.0E-35
60  CONTINUE
C      GO TO 40
C
C      APPLY NEWTON'S FORMULA
70  XNEW=X*(1.+RATIO+ZBAR*EXP(X-KP*ALOG(X)))
C
C      TEST FOR CONVERGENCE
C      IF(ABS(XNEW-X) .LE. EPS*XNEW) GO TO 80

```



```
      X=XNEW  
      GC TO 30  
C  
C      NEWTON'S METHOD CONVERGED.  SET FUNCTION VALUE  
C      AND EXIT.  
80  INVGAM=XNEW  
      RETURN  
      END
```



INVGAM	CSECT		
SAVE	EQU	2	REGISTER EQUATES
BASE	EQU	3	
REGA	EQU	4	
REGB	EQU	5	
REGC	EQU	6	
	STM	14,REGC,12(13)	SAVE GENERAL PURPOSE REGS
	LR	BASE,15	SET UP BASE REGISTER
	USING	INVGAM,BASE	
*			
*	LR	SAVE,13	SAVE CALLING PROGRAM SAVE
*	LA	13,SVAREA	AREA ADDRESS
	ST	13,8(0,SAVE)	PUT OWN SAVE AREA ADDRESS
	ST	SAVE,4(0,13)	IN REGISTER 13
	STD	2,SAVED	STORE SAVE AREA ADDRESSES
	STD	4,SAVED+8	SAVE FLOATING POINT REGS
	STD	6,SAVED+16	
*			
	L	REGB,4(1)	GET ADDRESS OF 1 - Z
	LE	0,0(REGB)	LOAD 1 - Z
	LTER	0,0	TEST IF 1 - Z > 0.0
	BC	2,T1	
	LE	0,=E'0.0'	Z .LE. 0; SET INVGAM TO 0
*			AND RETURN
*	LR	13,SAVE	RESTORE CALLING PROGRAM
	LM	2,REGC,28(13)	SAVE ADDRESS
	L	14,12(13)	RESTORE GP REGISTERS
	MVI	12(13),X'FF'	RESTORE RETURN ADDRESS
	BCR	14,15	SET END OF SUBROUTINE FLAG
			RETURN
*			
T1	STE	0,Z	SAVE VALUE OF 1 - Z
*			
	L	REGB,0(1)	GET ADDRESS OF K
	LE	0,0(REGB)	LOAD VALUE OF K
	CE	0,K	TEST IF VALUE IS NEW
	BC	8,T2	
	STE	0,K	VALUE IS NEW, COMPUTE NEW
	LCER	2,0	CONSTANTS.
	STE	2,KNEG	KNEG = -K
	AE	2,=E'1.0'	
	STE	2,KBAR	KBAR = 1 - K
*			
	LA	1,ARG1	GET GAMMA(K)
	L	15,=V(GAMMA)	
	BALR	14,15	
	STE	0,GK	GK = GAMMA(K)
*			
T2	LE	0,Z	COMPUTE Z-RELATED
	LCER	0,0	CONSTANTS
	ME	0,GK	
	STE	0,ZBAR	ZBAR = -GAMMA(K) * (1 - Z)
	LE	0,=E'1.0'	
	STE	0,X	X = 1
*			
*			
*			
LOOP1	LE	0,KBAR	INITIALIZE CONTINUED
*			FRACTION CONSTANTS
	STE	0,Y	Y = 1 - K
	LE	2,=E'1.0'	
	STE	2,V	V(1) = 1.0
	LE	4,X	
	STE	4,V+4	V(2) = X
	LER	6,4	
	AER	4,2	
	STE	4,V+8	V(3) = X + 1.0
	AER	0,4	
	STE	0,W	W = X + Y + 1.0
	MER	0,6	



```

STE      0,V+12      V(4) = W * X
DER      4,0
STE      4,PROB      PROB = V(3) / V(4)
STE      2,NUM        NUM = 1

**
**          CONTINUED FRACTION APPROXIMATION
**
LOOP2    LE      0,Y
          AE      0,=E'1.0'
          STE     0,Y      Y = Y + 1.0
          LE      4,W
          AE      4,=E'2.0'
          STE     4,W      W = W + 2.0
          MER     2,0

*
          LE      0,V+12
          MER     0,4
          LE      6,V+4
          MER     6,2
          SER     0,6
          STE     0,V+20      V(6) = V(4) * W -
                                V(2) * Y * NUM

*
          LE      0,V+8
          MER     0,4
          LE      6,V
          MER     6,2
          SER     0,6
          STE     0,V+16      V(5) = V(3) * W -
                                V(1) * Y * NUM

*
          LER     4,0

*
          DE      0,V+20      NEW RATIO = V(5) / V(6)
          LER     2,0
          SE      0,PROB      TEST FOR CONVERGENCE
          STE     2,PROB
          LPER    0,0
          ME      2,=E'1.0E-6'
          CER     0,2
          BC      12,CNVRG      ABS(RATIO - PROB) :
                                EPS * PROB
          MVC     V(16),V+8      SHIFT RATIO VECTOR
          CE      4,=E'1.0E35'      TEST IF SCALING NEEDED
          BC      12,INCMNT

*
          LE      0,=E'1.0E-35'      SCALE DOWN RATIO VECTOR TO
          LER     2,0                PREVENT OVERFLOW
          ME      0,V
          STE     0,V
          LER     0,2
          ME      0,V+4
          STE     0,V+4
          LER     0,2
          ME      0,V+8
          STE     0,V+8
          LER     0,2
          ME      0,V+12
          STE     0,V+12

*
INCMNT   LE      2,NUM      INCREMENT COUNT
          AE      2,=E'1.0'
          STE     2,NUM
          BC      15,LOOP2

**
**          CNVRG
**
          LA      1,ARG2      FIND ALOG(X)
          L       15,=V(ALOG)
          BALR    14,15

*
          ME      0,KNEG
          AE      0,X
          STE     0,XP      XP = -K * ALOG(X) + X

*
          LA      1,ARG3      FIND EXP(XP)

```





	L	15,=V(EXP)	
	BALR	14,15	
*			
	ME	0,ZBAR	$X_{NEW} = X * (1 + \text{RATIO} - \frac{X^{-K}}{(1-Z)*\text{GAMMA}(K)} E^X$
*			
	AE	0,PROB	(BASIC NEWTON-RAPHSON
	AE	0,=E'1.0'	RECURSION RELATION)
	ME	0,X	
	LER	2,0	
*			
	SE	2,X	TEST FOR NEWTON-RAPHSON
	LPER	2,2	CONVERGENCE
	STE	0,X	
	ME	0,=E'1.0E-6'	
	CER	2,0	ABS(XNEW - X) :
	BC	2,LOOP1	EPS * XNEW
*			
	LE	0,X	DONE. LOAD FUNCTION VALUE.
*			
	LD	2,SAVED	RESTORE FLOATING POINT
	LD	4,SAVED+8	REGISTERS
	LD	6,SAVED+16	
	LR	13,SAVE	RESET OLD SAVE AREA
	LM	2,REGC,28(13)	RESTORE OLD GP REGISTERS
	L	14,12(13)	RESTORE RETURN ADDRESS
	MVI	12(13),X'FF'	SET END OF SUBROUTINE FLAG
	BCR	15,14	RETURN
	DS	0D	
DATA			
*			
SAVED	DS	3D	
SVAREA	DS	18F	
*			
K	DC	4X'00'	
Z	DC	4X'00'	
*			
KNEG	DC	4X'00'	
KBAR	DC	4X'00'	
GK	DC	4X'00'	
ZBAR	DC	4X'00'	
*			
X	DC	4X'00'	
Y	DC	4X'00'	
V	DC	24X'00'	
W	DC	4X'00'	
PROB	DC	4X'00'	
NUM	DC	4X'00'	
XP	DC	4X'00'	
*			
ARG1	DC	X'80'	
	DC	AL3(K)	
*			
ARG2	DC	X'80'	
	DC	AL3(X)	
*			
ARG3	DC	X'80'	
	DC	AL3(XP)	
	END		



```

REAL*4    K,M2,MD
REAL*8    M,R,R1,DBLE
REAL*8    Y1,Y2
DIMENSION M(4),M2(4),MD(4,10)
DIMENSION K(4)
DIMENSION QUANT(23),Q2(23),QD(23,10),SCORE(23)
DIMENSION ALPHA(23),IQ(23)
DIMENSION S(10000),U(100)
DATA      NSAMPL/10000/,SAMPL/10000./
DATA      ALPHA/.001,.002,.005,.01,.02,.025,.05,.1,
*          .2,.3,.4,.5,.6,.7,.8,.9,.95,.975,.98,
*          .99,.995,.998,.999/
DATA      K/.1,.25,.50,.75/

```

```

C
C      INITIALIZE CONSTANTS
IX=13572183
DC 10 I=1,23
IQ(I)=SAMPL*ALPHA(I)+0.5
10 CONTINUE
N=30

C
C      START SIMULATION LOOP
DC 160 INDEX=1,4

C
C      INITIALIZE GAMMA GENERATOR
CALL START(K(INDEX),1.,IX)

C
C      FIND THEORETICAL MEAN AND VARIANCE
TMEAN=(FLOAT(N)+1.)*.5
TVAR=(FLOAT(N)**2-1.)/(12.*(K(INDEX)*FLOAT(N)+1.))
TSD=SQRT(TVAR)

C
C      ZERO OUT DATA COLLECTION ARRAYS
DC 20 I=1,4
M(I)=0.
M2(I)=0.
20 CONTINUE
DC 30 I=1,23
QUANT(I)=0.
30 CONTINUE

C
C      PERFORM 10 REPLICATIONS OF THE SIMULATION
DC 90 L=1,10

C
C      GENERATE 10,000 RATIO STATISTICS
DO 60 J=1,NSAMPL
Y1=0.
Y2=0.
DC 40 I=1,N
Y1=Y1+DBLE(GAMA(IX))
Y2=Y2+Y1
40 CONTINUE

C
C      FIND RATIO
R=Y2/Y1
S(J)=R
R1=R

C
C      ACCUMULATE MOMENT ESTIMATORS
DC 50 I=1,4
M(I)=M(I)+R1
R1=R1*R
50 CONTINUE
60 CONTINUE

C
C      DETERMINE MOMENT ESTIMATES FOR CURRENT SAMPLE
DC 70 I=1,4
MD(I,L)=M(I)/SAMPL
M2(I)=M2(I)+MD(I,L)
M(I)=0.
70 CONTINUE

```



```

C      SORT SAMPLE AND GET QUANTILE ESTIMATES
      CALL SORT(S,1,NSAMPL)
      DO 80 I=1,23
      QD(I,1)=S(IQ(I))
      QUANT(I)=QUANT(I)+S(IQ(I))
80    CCNTINUE
90    CONTINUE

C      FIND MEAN AND VARIANCE OF THE 10 MOMENT ESTIMATES
      DO 110 I=1,4
      M(I)=M2(I)/10.
      M2(I)=0.
      DO 100 J=1,10
      M2(I)=M2(I)+(M(I)-MD(I,J))**2
100   CCNTINUE
      M2(I)=M2(I)/9.
110   CCNTINUE

C      FIND MEAN AND VARIANCE OF THE 10 QUANTILE ESTIMATES
      DO 130 I=1,23
      QUANT(I)=QUANT(I)/10.
      Q2(I)=0.
      DO 120 J=1,10
      Q2(I)=Q2(I)+(QUANT(I)-QD(I,J))**2
120   CCNTINUE
      Q2(I)=Q2(I)/9.
      SCORE(I)=(QUANT(I)-TMEAN)/TSD
130   CONTINUE

C      CCMPUTE SAMPLE STATISTICS
      YMEAN=M(1)
      VAR= SAMPL*(M(2)-M(1)*M(1))/(SAMPL-1.)
      SD=SQRT(VAR)
      SKEW=(M(3)-(2.*VAR+M(2))*M(1))/(SD*VAR)
      CKURT=(((-3.*M(1)*M(1)+6.*M(2))*M(1)-4.*M(3))
      *      *M(1)+M(4))/(VAR*VAR)

C      PRINT OUT RESULTS
      WRITE(6,140)K(INDEX),N,M,M2,YMEAN,VAR,SD,SKEW,CKURT
140   FORMAT('1SIMULATION RESULTS FOR K=',F5.2,' AND SAMPL',
      *      'E SIZE =',I4
      *      '///'OTHE FIRST FOUR MOMENTS ARE:'//10X1P4E19.6/
      *      'VARIANCE:',4E19.6///'OMEAN      =' ,E14.6/
      *      'OVARIANCE =',E14.6,4X,'STANDARD DEVIATION=',
      *      E14.6/'OSKEWNESS =' ,E14.6/'OKURTOSIS =' ,E14.6/)

C      WRITE(6,150){ALPHA(I),QUANT(I),Q2(I),SCORE(I),I=1,23}
150   FORMAT('///'OQUANTILES OF THE DISTRIBUTION:'
      *      '///5X,'ALPHA',6X,'ALPHA-QUANTILE',8X,'VARIANCE'
      *      ',3X,'NORMALIZED QUANTILE'//
      *      {3XOPF7.3,1P2E19.6,OPF14.6})
160   CCNTINUE
      STOP
      END

```



```

SUBROUTINE SCRT(A,II,JJ)
DIMENSION A(JJ),IL(16),IL(16)
INTEGER*4 A,T,TT
M=1
I=II
J=JJ
5 IF(I .GE. J) GO TO 70
10 K=I
IJ=(I+J)/2
T=A(IJ)
IF(A(I) .LE. T) GO TO 20
A(IJ)=A(I)
A(I)=T
T=A(IJ)
20 L=J
IF(A(J) .GE. T) GO TO 40
A(IJ)=A(J)
A(J)=T
T=A(IJ)
IF(A(I) .LE. T) GO TO 40
A(IJ)=A(I)
A(I)=T
T=A(IJ)
GO TO 40
30 A(L)=A(K)
A(K)=TT
40 L=L-1
IF(A(L) .GT. T) GO TO 40
TT=A(L)
50 K=K+1
IF(A(K) .LT. T) GO TO 50
IF(K .LE. L) GO TO 30
IF(L-I .LE. J-K) GO TO 60
IL(M)=I
IU(M)=L
I=K
M=M+1
GO TO 80
60 IL(M)=K
IU(M)=J
J=L
M=M+1
GO TO 30
70 M=M-1
IF(M .EQ. 0) RETURN
I=IL(M)
J=IU(M)
80 IF(J-I .GE. II) GO TO 10
IF(I .EQ. II) GO TO 5
I=I-1
90 I=I+1
IF(I .EQ. J) GO TO 70
T=A(I+1)
IF(A(I) .LE. T) GO TO 90
K=I
100 A(K+1)=A(K)
K=K-1
IF(T .LT. A(K)) GO TO 100
A(K+1)=T
GO TO 90
END

```





# BIBLIOGRAPHY

- 1 Abramowitz, M. and Stegun, I.A., (ed.), Handbook of Mathematical Functions, pp. 260-263, Dover, 1964.
- 2 Ahrens, J.H. and Dieter, U., "Computer Methods for Sampling from the Exponential and Normal Distributions," Communications of the ACM, v. 15, pp. 873-882, October 1972.
- 3 Anderson, T.W. and Darling, D.A., "A Test of Goodness of Fit," Journal of the American Statistical Association, v. 49, pp. 765-769, December 1954.
- 4 Bartholomew, D.J., "Tests for Randomness in a Series of Events when the Alternative is a Trend," Journal of the Royal Statistical Society B, v. 18, pp. 234-239, 1956.
- 5 Cox, D.R., "Some Statistical Methods Connected with Series of Events," Journal of the Royal Statistical Society B, v. 17, pp. 129-164, 1955.
- 6 Cox, D.R. and Lewis, P.A.W., The Statistical Analysis of Series of Events, Methuen, 1966.
- 7 Cox, D.R., "Statistical Analysis of Dependencies in Point Processes," in Stochastic Point Processes, edited by P.A.W. Lewis, pp. 55-66, Wiley, 1972.
- 8 Cramer, H., Mathematical Methods of Statistics, Princeton University Press, 1946.
- 9 Fox, B.L., "Remarks about Random Sampling from Beta and Gamma Distributions," Notes on Operations Research - 5, ORC 66-13, University of California, Berkeley, April 1966.
- 10 Gnedenko, B.V., The Theory of Probability, Third ed., Chelsea, 1962.
- 11 Goodman, A.S., Lewis, P.A.W. and Robbins, H.E., "Simultaneous Estimation of Large Numbers of Quantiles in Simulation Experiments," to be published in Communications in Statistics, 1973.
- 12 IBM Research Report RC-1094, Statistical Distribution Programs for a Computer Language, by R.E. Bargmann and S.P. Ghosh, 22 December 1963.
- 13 Johnk, M.D., "Erzeugung von Betaverteilten und Gammaverteilten Zufallszahlen (Generation of Beta-distributed and Gamma-distributed Random Numbers)," Metrika, v. 8, pp. 5-15, 1964.



- 14 Knuth, D.E., The Art of Computer Programming, Volume II: Seminumerical Algorithms, Addison-Wesley, 1969.
- 15 Knuth, D.E., The Art of Computer Programming, Volume III: Sorting and Searching, Addison-Wesley, 1972.
- 16 Laha, R.G., "On a Characterisation of the Gamma Distribution," Annals of Mathematical Statistics, v. 25, pp. 784-787, December 1954.
- 17 Lewis, P.A.W., "Distribution of the Anderson-Darling Statistic," Annals of Mathematical Statistics, v. 32, pp. 1118-1124, December 1961.
- 18 Lewis, P.A.W., "Some Results on Tests for Poisson Processes," Biometrika, v. 52, pp. 67-78, June 1965.
- 19 Lewis, P.A.W., Goodman, A.S. and Miller, J.M., "A Pseudo-Random Number Generator for the System/360," IBM Systems Journal, No. 2, pp. 136-146, 1969.
- 20 Lewis, P.A.W. and Shedler, G.S., "Empirically Derived Micromodels for Sequences of Page Exceptions," IBM Journal of Research and Development, v. 17, pp. 86-100, March 1973.
- 21 Lukacs, E., "A Characterization of the Gamma Distribution," Annals of Mathematical Statistics, v. 26, pp. 319-324, June 1955.
- 22 Marsaglia, G., MacLaren, M.D. and Bray, T.A., "A Fast Procedure for Generating Exponential Random Variables," Communications of the ACM, v. 7, pp. 298-300, May 1964.
- 23 Marsaglia, G., MacLaren, M.D. and Bray, T.A., "A Fast Procedure for Generating Normal Random Variables," Communications of the ACM, v. 7, pp. 4-10, January 1964.
- 24 Naval Postgraduate School Technical Report NPS55LW72111A, Large Scale Computer-Aided Statistical Mathematics, by P.A.W. Lewis, November 1972.
- 25 Naylor, T.H., Balintfy, J.L., Burdick, D.S. and Chu, K., Computer Simulation Techniques, pp. 87-89, Wiley, 1968.
- 26 Phillips, D.T. and Beightler, C., "A Procedure for Generating Gamma Variates with Non-Integer Parameter Sets," Proceedings of the Fifth Annual Conference on Applications of Simulation, Boston, 1977.
- 27 Singleton, R.C., "Algorithm 347: An Efficient Algorithm for Sorting with Minimal Storage," Communications of the ACM, v. 12, p. 185, March



1969.

- 28 United States Air Force Project Rand Report R-641-PR, Generating Gamma Distributed Variates for Computer Simulation Models, by M.B. Berman, The Rand Corporation, February 1977.
- 29 Wald, A., and Wolfowitz, J., "Statistical Tests Based on Permutations of the Observations," Annals of Mathematical Statistics, v. 15, 1944, pp. 358-372.



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## 13. ABSTRACT

In testing the hypothesis that there is no monotone trend in a gamma renewal process, the use of the statistic

$$Y_J = Y_{2J} / Y_{1J},$$

where

$$Y_{1J} = \sum_{i=1}^J X_i$$

and

$$Y_{2J} = \sum_{i=1}^J S_i,$$

is investigated. The mean and variance of  $Y_J$  is developed as a function of  $J$  and it is shown that  $Y_J$  is asymptotically normal as  $J \rightarrow \infty$  for the gamma renewal process. A high-speed, theoretically exact gamma pseudo-random variate generator is developed, tested and compared with other known techniques. The generator is then used to obtain the distribution of  $Y_J$  through digital computer simulation for small and moderate values of  $J$ .



## KEY WORDS

## LINK A

## LINK B

## LINK C

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Random Number Generators

Gamma Distribution

Gamma Random Variates

Gamma Renewal Process

Trend Tests

Statistical Simulation Techniques



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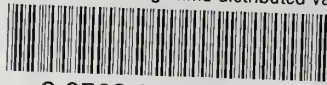
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